Where do Complex Numbers come from?

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1 Introduction.

By far, as a freshman, you have been doing math in the "real" world: $\mathbb{R}$ (lines), $\mathbb{R}^2$ (planes), etc. We entered into the mathematical world by studying our fingers, from which the notion of "counting" crystallizes. We were told how to do additions and multiplications, and later subtractions and divisions, their inverses. More types of numbers were introduced, accompanied by many more operations, further confounding our naive minds. What's the point of all these fabrications?

But it was not until you were taught not to take the square root of negative numbers that things seemed to leap out of the sensible world. Performing this "illegal" operation should be much wilder than dividing by zero (which, by intuition, equals infinity, but what is "infinity"?) The simplest example is to find a solution to the equation

$$x^2 + 1 = 0.$$

This problem has been bothering humanity, especially mathematicians in their study of polynomial equations, for millennia. So it is natural for us to trip over the same trunk.

This is the story of the origin of complex numbers. The equation $x^2 + 1 = 0$ itself gives rise to the first and most crucial non-real number i. This accolade is shared by the quadratic polynomial $p(x) = x^2 + 1$, which is later called the "minimal polynomial" of i over $\mathbb{R}$. But don't bother with the technical terms as a beginner - it will merge into your language seamlessly after years of exposure and practice. Mathematicians are just those who think a bit longer and deeper than the average person on these matters.

Some of you may have already burst out unsatisfied remarks: where does i come from? It is less of a (real) number and more of a variable. This is a wonderful question that touches the heart of many math problems. That is, do we really need to define something solid to study this object? By diving deeper and deeper into the theoretical world, you will witness a gradual transformation from yes to no towards this question, as what we are studying are not objects themselves but their interactions with other objects. The current study of math from this perspective is the field of category theory, whose notations are (sadly) ENTIRELY inaccessible to any normal individual.

Rattling on all that I have said, it is high time to provide a solid construction of complex numbers from what we know. Though by reading through the following construction for the first time, you may still discredit its validity, after a careful perusal, PLUS some deliberation, you could most likely see the light at the end of the tunnel and feel some empathy for the mathematicians.

2 A Construction of $\mathbb{C}$.

Definition (Equivalence Relation). Given a non-empty set S, an *equivalence relation* R on S is a subset of $S \times S$ such that for all $x, y, z \in S$,

- Reflexivity: xRx;
- Symmetry: $xRy = yRx$;
- Transitivity: $xRy, yRz \Rightarrow xRz$.

Remark. We write xRy if and only if the pair $(x, y) \in R$. Classical equivalence relations include equalities of all objects, similarity & congruence of triangles, etc.

The easiest and most common way to understand equivalence relations is a partition of the set S into "equivalence classes". Any two elements from the same chunk of S are equivalent, while elements from different chunks are not equivalent. We often write $x \sim y$ instead of xRy for clarity, and denote the equivalence class of x by $[x]$. x is called a representative of $[x]$.

Definition (Generic Complex Numbers). Let $\mathbb{R}[x]$ be the set of all real coefficient polynomials in x. Let $f(x), g(x) \in \mathbb{R}[x]$ be equivalent if and only if $p(x) | (f(x) - g(x))$ (or $f(x) \equiv g(x) \pmod{p(x)}$). Let $\mathbb{C}$ be the set of all equivalence classes of $\mathbb{R}[x]$ under this equivalence relation.

Remark. You should check that this is a well-defined equivalence relation. After checking, it is safe to write, for instance, $1 \sim x^2+2 \sim x^3+x+1$, so $[1] = [x^2+2] = [x^3+x+1]$.

Given any $f(x) \in \mathbb{R}[x]$, we can apply the long division algorithm, divide $f(x)$ by $p(x)$, and yield a residue $r(x)$ whose degree must be strictly less than that of $p(x)$. So every equivalence class in $\mathbb{C}$ has a representative that takes the form $a + bx$ for some $a, b \in \mathbb{R}$.

Definition (Operations on $\mathbb{C}$). Define

$$[f(x)] \pm [g(x)] := [f(x) \pm g(x)], \quad [f(x)] \cdot [g(x)] := [f(x)g(x)].$$

Again, whenever utilizing representatives, you should readily check well-definedness, that the statement doesn't depend on the choice of representatives. What is less obvious is that every non-zero element $[f(x)] \in \mathbb{C}^* := \mathbb{C} \setminus \{[0]\}$ has a multiplicative inverse. Without loss of generality, we take $f(x) = a + bx$ where $a^2 + b^2 > 0$ and $\widetilde{f}(x) = a - bx$, then $f(x)\widetilde{f}(x) = a^2 - b^2x^2 \sim a^2 + b^2$, so indeed $[\frac{1}{a^2+b^2}\widetilde{f}(x)]$ is the inverse of $[f(x)]$:

$$[\frac{1}{a^2+b^2}\widetilde{f}(x)] \cdot [f(x)] = [f(x)] \cdot [\frac{1}{a^2+b^2}\widetilde{f}(x)] = [1],$$

where $[1]$ serves as the multiplicative identity. You shall also be notified that by defining addition and subtraction above simultaneously, we have implicitly conceded that $[0]$ is the additive identity:

$$[-f(x)] + [f(x)] = [f(x)] + [-f(x)] = [0].$$

Definition (Complex Numbers). With all four basic arithmetic operations $+-\times\div$ available on $\mathbb{C}$, we successfully construct the set of complex numbers. The mapping $\iota:\mathbb{R}\hookrightarrow\mathbb{C},\quad r\mapsto[r]$ shows that $\mathbb{C}$ encodes all arithmetic information of $\mathbb{R}$. But there is $\widetilde{\iota}:\mathbb{R}\hookrightarrow\mathbb{C},\quad r\mapsto[rx]$, which is another effective embedding, which betrays (infinitely) many ways to encode $\mathbb{R}$ into $\mathbb{C}$. These two mappings or their images are called the *real axis* and *imaginary axis* of $\mathbb{C}$ respectively.

By this definition, there is a canonical identification between $\mathbb{C}$ and the plane $\mathbb{R}^2$, $[a+bx]\mapsto(a,b)\in\mathbb{R}^2$. It provides nice geometry not only because we bump into the form $a+bx$ that resembles a point. For one thing, linear operations of complex numbers coincide with linear vector operations on $\mathbb{R}^2$; for another, complex multiplications are proven to be rotations and dilations with respect to the origin, having profound implications such as the famous "Euler's identity":

$$e^{i\pi}+1=0,$$

where the exponential is still undefined so far.

For convenience, we always write $a+bi$ instead of $[a+bx]$, because $[a+bx]=[a]+[b]\cdot[x]$. So, you got it: the equivalence class $[x]$ of the free variable x is the very i we have long been craving for!

3 Afterword.

It is my pleasure to take you on a tour into the field of abstract algebra, the modern version of the ancient subject "algebra". The monumental turning point must be Galois's godsend creation of group theory 200 years ago. From then on, the goal for mathematicians is not simply solving high-degree polynomial equations, but to study algebraic structures with a bird's-eye view, like what we've been doing here. Although present-day algebra looks pretty like a neat copy of the scratch paper of a lunatic, you should always believe that any such fantasy originates from the physical world and is rooted in our inherent intuitions of logic.

Returning to our topic, complex numbers. There are, of course, other constructions available once you learn some linear algebra. The set of matrices

$$\mathbb{C}:=\left\{\begin{pmatrix}a & -b\\ b & a\end{pmatrix}:a,b\in\mathbb{R}\right\}$$

together with the standard matrix addition and multiplication also results in the same complex structure, and it is your job to explore more approaches if you are significantly inspired and motivated by my dedicated pedagogy.

We used to believe that complex numbers didn't exist, but they are everywhere in our daily lives. From signal processing to machine learning, there are always arenas for complex numbers to flex their muscles, making life easier and even saving lives. This is just one single concept, and the bridges between a giant store of them to our world are yet to be discovered.