Two Trigonometric Integrals

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April 2026

It is among the first nontrivial examples after introducing the technique of integration by parts to compute the polynomial-type trigonometric integrals

$$I(m,n) = \int \cos^m x \sin^n x \,\mathrm{d}x, \quad J(m,n) = \int \tan^m x \sec^n x \,\mathrm{d}x,$$

where $m, n \in \mathbb{N}$. These integrals are homogeneous in nature, both of which are of the form

$$\int P(f(x), g(x)) \,\mathrm{d}x, \quad P(a,b) = a^m b^n,$$

where the two pairs of functions $(\cos, \sin)$ and $(\tan, \sec)$ share analogous identities and derivatives:

$$\sin^2 x + \cos^2 x = 1, \quad (\sin x)' = \cos x, \quad (\cos x)' = -\sin x,$$

$$\sec^2 x - \tan^2 x = 1, \quad (\sec x)' = \tan x \sec x, \quad (\tan x)' = \sec^2 x.$$

Now we compute the induction formula for $I(m,n)$,

$$\begin{aligned}
I(m,n) &= \int \cos^m x \sin^n x \,\mathrm{d}x \\
&= -\int \cos^m x \sin^{n-1} x \,\mathrm{d}(\cos x) \\
&= -\cos^{m+1} x \sin^{n-1} x + \int \cos x \,\mathrm{d}(\cos^m x \sin^{n-1} x) \\
&= -\cos^{m+1} x \sin^{n-1} x + \int \cos x (-m \cos^{m-1} x \sin^n x + (n-1) \cos^{m+1} x \sin^{n-2} x) \,\mathrm{d}x \\
&= -\cos^{m+1} x \sin^{n-1} x - mI(m,n) + (n-1) I(m+2, n-2).
\end{aligned}$$

Similarly, for $J(m,n)$,

$$\begin{aligned}
J(m,n) &= \int \tan^m x \sec^n x \,\mathrm{d}x \\
&= \int \tan^m x \sec^{n-2} x \,\mathrm{d}(\tan x) \\
&= \tan^{m+1} x \sec^{n-2} x - \int \tan x \,\mathrm{d}(\tan^m x \sec^{n-2} x) \\
&= \tan^{m+1} x \sec^{n-2} x - \int \tan x (m \tan^{m-1} x \sec^n x + (n-2) \tan^{m+1} x \sec^{n-2} x) \,\mathrm{d}x \\
&= \tan^{m+1} x \sec^{n-2} x - mJ(m,n) - (n-2) J(m+2, n-2).
\end{aligned}$$

The integrals can be drastically simplified if n is odd in $I(m,n)$ or n is even in $J(m,n)$:

$$\sin^n x \,\mathrm{d}x = -(1 - \cos^2 x)^{\frac{n-1}{2}} \,\mathrm{d}(\cos x), \quad \sec^n x \,\mathrm{d}x = (\tan^2 x + 1)^{\frac{n-2}{2}} \,\mathrm{d}(\tan x),$$

so that the integrands are reduced to polynomials by substitution of variables. By the symmetry of the pairs $(\cos, \sin)$ and $(\tan, \sec)$, similar reductions are available if m is odd in both $I(m,n)$ and $J(m,n)$:

$$\cos^m x \,\mathrm{d}x = (1 - \sin^2 x)^{\frac{m-1}{2}} \,\mathrm{d}(\sin x), \quad \tan^m x \,\mathrm{d}x = \frac{(\sec^2 x - 1)^{\frac{m-1}{2}}}{\sec x} \,\mathrm{d}(\sec x).$$

It remains to compute the cases $I(2k, 2l)$ and $J(2k, 2l+1)$, and by applying their induction formulae, it remains to compute the base case $k = 0$. Now we compute the induction formula for $I(0, 2l)$,

$$\begin{aligned}
I(0,2l) &= -\cos x \sin^{2l-1} x + (2l-1) I(2, 2l-2) \\
&= -\cos x \sin^{2l-1} x + (2l-1) \int (1 - \sin^2 x) \sin^{2l-2} x \, \mathrm{d}x \\
&= -\cos x \sin^{2l-1} x + (2l-1) I(0, 2l-2) - (2l-1) I(0, 2l).
\end{aligned}$$

Similarly, for $J(0, 2l+1)$,

$$\begin{aligned}
J(0,2l+1) &= \tan x \sec^{2l-1} x - (2l-1) J(2, 2l-1) \\
&= \tan x \sec^{2l-1} x + (2l-1) \int (1 - \sec^2 x) \sec^{2l-1} x \, \mathrm{d}x \\
&= \tan x \sec^{2l-1} x + (2l-1) J(0, 2l-1) - (2l-1) J(0, 2l+1).
\end{aligned}$$

Applying these induction formulae, it remains to compute the base case $k = l = 0$, where

$$I(0,0) = x, \quad J(0,1) = \log |\tan x + \sec x| .$$

The induction formulae for $I(0, 2l)$ and $J(0, 2l+1)$ can be written as

$$\begin{aligned}
I(0,2l) &= \frac{2l-1}{2l} I(0, 2l-2) - \cos x \, \frac{\sin^{2l-1} x}{2l}, \\
J(0,2l+1) &= \frac{2l-1}{2l} J(0, 2l-1) + \tan x \, \frac{\sec^{2l-1} x}{2l},
\end{aligned}$$

from which we derive their closed-form expressions:

$$\begin{aligned}
I(0,2l) &= a_l \left(I(0,0) - \cos x \sum_{j=1}^{l} b_j \sin^{2j-1} x \right), \\
J(0,2l+1) &= a_l \left(J(0,1) + \tan x \sum_{j=1}^{l} b_j \sec^{2j-1} x \right),
\end{aligned}$$

where

$$a_l = \frac{(2l-1)!!}{(2l)!!}, \quad b_j = \frac{(2j-2)!!}{(2j-1)!!}.$$

Similarly, the induction formulae for $I(2k, 2l)$ and $J(2k, 2l+1)$ can be written as

$$\begin{aligned}
I(2k,2l) &= \frac{2k-1}{2l+1} I(2k-2, 2l+2) + \frac{\cos^{2k-1} x \sin^{2l+1} x}{2l+1}, \\
J(2k,2l+1) &= -\frac{2k-1}{2l+1} J(2k-2, 2l+3) + \frac{\tan^{2k-1} x \sec^{2l+1} x}{2l+1},
\end{aligned}$$

from which we derive their closed-form expressions:

$$\begin{aligned}
I(2k,2l) &= c_{kl} I(0, 2k+2l) + \sum_{j=0}^{k-1} d_{kl}^{(j)} \cos^{2j+1} x \sin^{2k+2l-2j-1} x, \\
J(2k,2l+1) &= (-1)^k c_{kl} J(0, 2k+2l+1) + \sum_{j=0}^{k-1} (-1)^{k+j+1} d_{kl}^{(j)} \tan^{2j+1} x \sec^{2k+2l-2j-1} x,
\end{aligned}$$

where

$$c_{kl} = \frac{(2k-1)!!(2l-1)!!}{(2k+2l-1)!!}, \quad d_{kl}^{(j)} = \frac{(2k-1)!!(2l-1)!!}{(2j+1)!!(2k+2l-2j-1)!!}.$$

Constant C is omitted throughout the computation.

In practice, the induction formulae are more useful, as closed-form expressions often do not exist for more general integrals.