The Method of Infinitesimal Analysis

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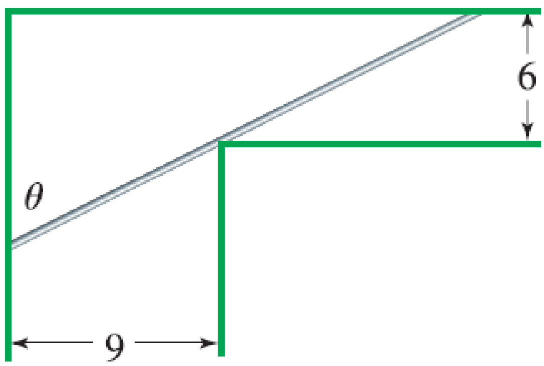
1 A Homework Problem Revisited.

This is extracted from "Math 124 Section 4.7 Homework, Question 11."

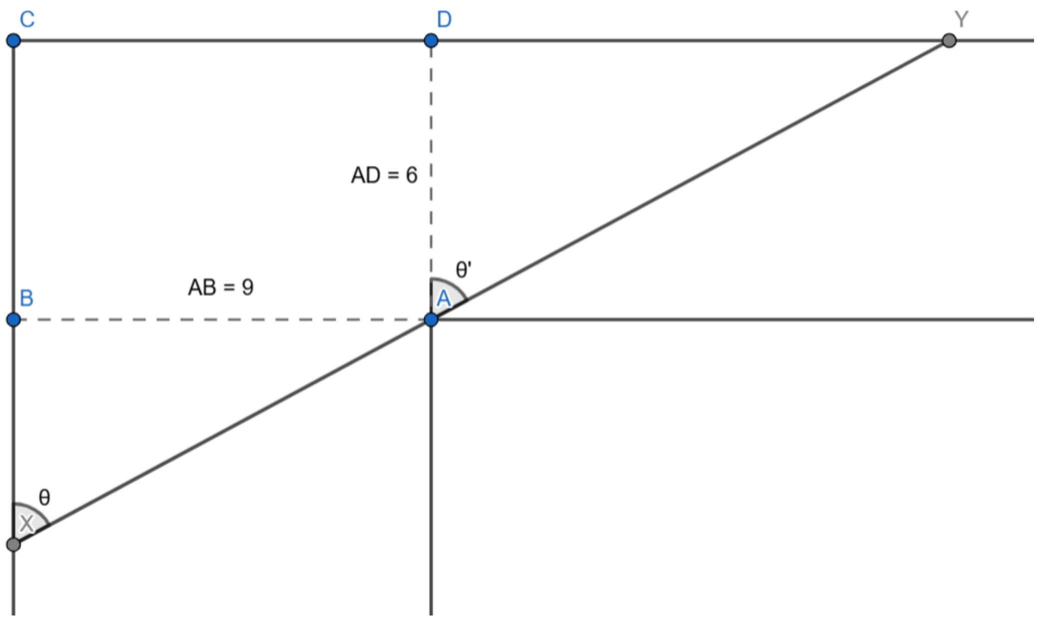
> A steel pipe is being carried down a hallway that is 9 ft wide. At the end of the hall, there is a right-angled turn into a narrower hallway, 6 ft wide. What is the length of the longest pipe (in ft) that can be carried horizontally around the corner? (Round your answer to one decimal place.)



The figure represents a top-down view of the hallway and the steel pipe. The hallway is represented by two vertical segments, each connected to two horizontal segments, and the steel pipe is represented by a diagonal line segment.

- The left and top hallway segment goes vertically upward, reaches a corner, turns, and goes horizontally to the right. The right and bottom hallway segment begins 9 units to the right of the left hallway segment, goes vertically upward, reaches a corner, and turns to go horizontally to the right, 6 units below the top hallway segment.
- A diagonal segment representing the pipe extends from the leftmost vertical hallway segment, goes up and to the right, touches the corner of the right and bottom hallway segments, continues up and to the right, and ends on the top hallway segment. The acute angle between the pipe segment and the leftmost hallway segment is labeled θ.

Remark. This problem may appear to ask for a maximum, but it actually asks for the maximum *tolerable* pipe length, which is equivalently the minimum length of the "space" formed by the corner.



The classic approach is to draw the perpendiculars AB and AD. Then $|AB| = 9$, $|AD| = 6$, and $\angle AXB = \angle YAD = \theta$. In $\triangle ABX$ and $\triangle YDA$, we have

$$\sin\theta = \frac{|AB|}{|AX|}, \qquad \cos\theta = \frac{|AD|}{|AY|}.$$

Hence

$$|XY| = |AX| + |AY| = \frac{9}{\sin\theta} + \frac{6}{\cos\theta}, \qquad \theta \in \left(0, \frac{\pi}{2}\right).$$

So one may set $f(\theta) = \dfrac{9}{\sin\theta} + \dfrac{6}{\cos\theta}$ and minimize it by taking the derivative and finding the critical point.

For adamant inequality enthusiasts, here is an approach using Hölder's inequality. Taking

$$a_1 = \frac{9}{\sin\theta}, \quad b_1 = \frac{6}{\cos\theta}, \quad a_2 = \sin^2\theta, \quad b_2 = \cos^2\theta$$

in the inequality

$$\sqrt[3]{(a_1 + b_1)^2 (a_2 + b_2)} \geq \sqrt[3]{a_1^2 a_2} + \sqrt[3]{b_1^2 b_2}, \quad a_i, b_j > 0,$$

we have a jaw-dropping one-line proof of this problem:

$$f(\theta)^2 = \left(\frac{9}{\sin\theta} + \frac{6}{\cos\theta}\right)^2 \left(\sin^2\theta + \cos^2\theta\right) \geq \left(9^{\frac{2}{3}} + 6^{\frac{2}{3}}\right)^3,$$

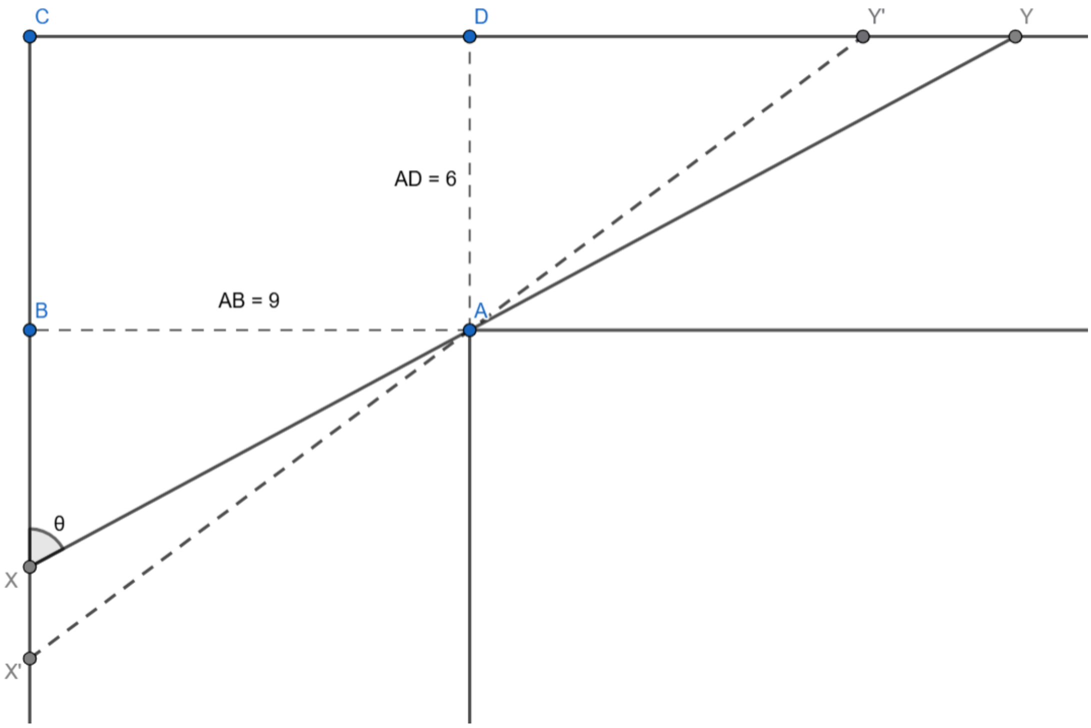
where equality is achieved when the vectors (a_1, b_1) and (a_2, b_2) are colinear.

2 The Infinitesimal Method.

There is an extremely visual method, attributed to Newton, which has been widely embraced by physicists and distrusted by mathematicians over the past four centuries. The idea originates with Democritus and Zeno in ancient Greece, over 2,400 years ago, and was formalized by Newton and Leibniz two millennia later.

The idea is to use the finitely small to approximate the infinitely small, or infinitesimals. Numerous rules were established by consensus, which, from a modern perspective, are too tenuous without the support of calculus. We will soon see how these empirical, rule-of-thumb results are informally used and formally interpreted in the following argument.

Recall that we are finding the local minimum of $|XY|$, the point where its increment or decrement is negligible. Under an infinitesimal perturbation, the length $|XY|$ remains approximately constant.



This means the increment of $|AX|$ is approximately equal to the decrement of $|AY|$. The phrase "approximately equal" is vague, but quantitatively the difference between these two perturbations, $\delta\,|AX|$ and $\delta\,|AY|$, is an *infinitesimal of higher order* than either of them. Here δ denotes the (absolute) infinitesimal operator.

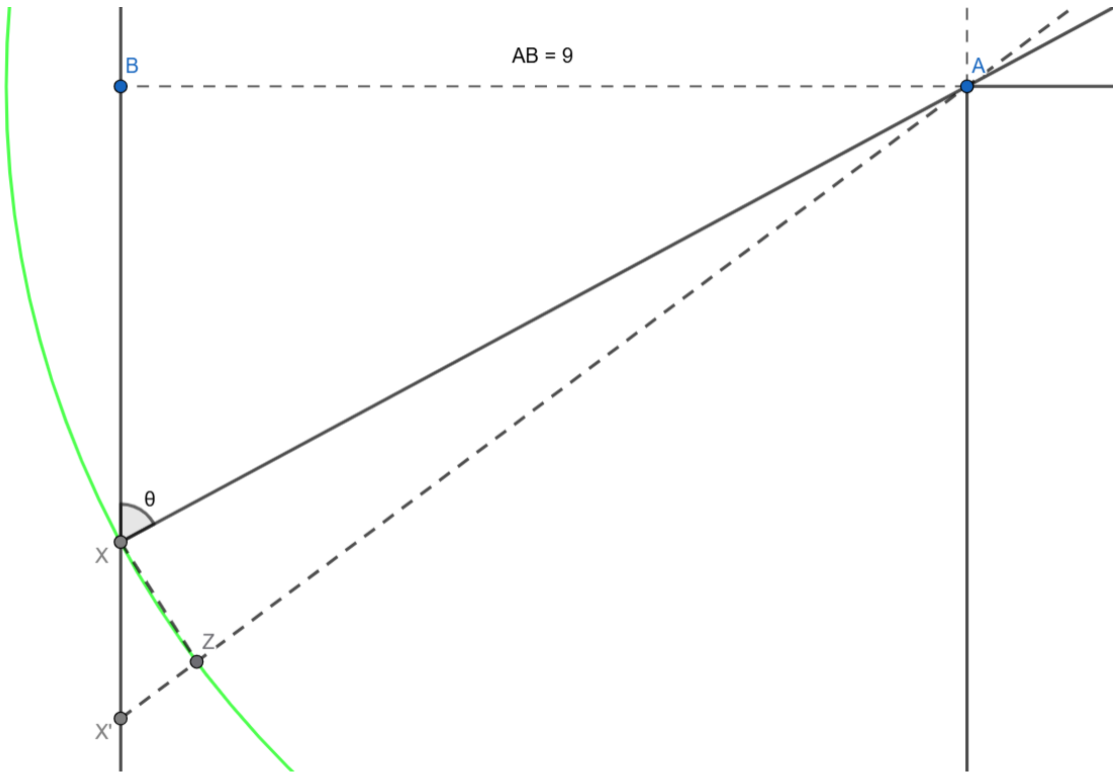
Now focus on visualizing $\delta\,|AX|$, the heart of the method. Consider $\delta\,|AX|$. Take the unique point Z on the segment AX' with $|AX| = |AZ|$ and connect X to Z. Since the angle $\angle XAX' = \delta\theta$ is tiny, $\triangle XZX'$ is nearly a right triangle, with

$$\angle XZX' \approx \pi/2, \quad \angle XX'Z \approx \theta.$$

In $\triangle AXZ$, the length $|XZ|$ is nearly equal to the arc XZ of the circle centered at A through X, Z, whose length is $|AX|\,\delta\theta$. Summarizing, we obtain the following estimate (try to appreciate its geometric plausibility):

$$\delta\,|AX| = |AX'| - |AX| = |X'Z| \approx |XZ| \cot\theta \approx |AX|\,\delta\theta \cot\theta = \frac{9\cos\theta}{\sin^2\theta}\,\delta\theta.$$

In mathematics, we would like to formalize δ and replace "$\approx$" with strict equality, but infinitesimal calculus of this type predates such abstraction.



It is time to apply the same estimate to $\delta\,|AY|$. All geometry is the same except that 9 becomes 6 and θ is replaced by its complement $\pi/2-\theta$:

$$\delta\,|AY| = \frac{6\sin\theta}{\cos^2\theta}\,\delta\theta.$$

We have asserted without proof that, when $|XY|$ is minimal, $\delta\,|AX| - \delta\,|AY|$ is of higher order than either perturbation, each of which is of order $\sim\delta\theta$. Therefore the difference

$$\delta\,|AX| - \delta\,|AY| = \left(\frac{9\cos\theta}{\sin^2\theta} - \frac{6\sin\theta}{\cos^2\theta}\right)\delta\theta \ll \delta\theta.$$

This is true only if, at the critical number of θ:

$$\frac{9\cos\theta}{\sin^2\theta} = \frac{6\sin\theta}{\cos^2\theta} \qquad \Longrightarrow \qquad \tan\theta = \sqrt[3]{\frac{3}{2}}.$$

3 Some Considerations.

At this point, any beginner in infinitesimal analysis might ask, "Why does all this make sense?" Mathematicians have been more troubled by this question than we are, of course. Although it is too much to fully formalize this entire procedure here, we can at least think of it in the following way.

We should have written all δ as Δ, indicating visibly large perturbations. Whenever we write $M \approx N$ for quantities M and N, it means that $M/N \to 1$ as the perturbation goes to zero. This defines an equivalence relation $\sim$ on these quantities. Under identification via $\sim$, we are justified in writing "=" instead of "$\approx$."

To master the infinitesimal method, it is more important to remember the geometry than the algebra. The challenging practice problems below will strengthen your understanding of this intriguing and enduring tool of math and physics.

4 Practice Problems.

1. Use the unit circle $x^2 + y^2 = 1$ and the vertical line $x = 1$ to visualize the relations among $\sin\theta$, θ, and $\tan\theta$. Show that as $\theta \to 0^+$,

$$\sin\theta < \theta < \tan\theta, \qquad \sin\theta \sim \theta \sim \tan\theta.$$

   What about other trigonometric functions? Discover more results of this kind.

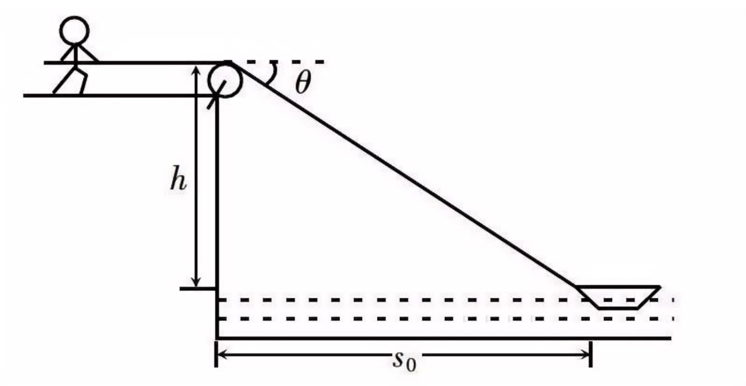
2. Show that the derivative of the inverse tangent function is $1/(x^2+1)$ using a triangle. Find more derivative formulas in this manner.

3. If A and B are two points on the same side of a line ℓ in a plane and $P \in \ell$, show that $\theta = \angle APB$ is locally maximized precisely when the circle through A, P, B is tangent to ℓ. Discuss the number of local maxima for different locations of A, B.

4. Use Fermat's principle of least time to prove Snell's law, or the *law of refraction.*

> The law states that, for a given pair of media, the ratio of the sines of the angle of incidence θ_1 and the angle of refraction θ_2 is equal to the refractive index of the second medium with respect to the first $n_{2,1}$, which is equal to the ratio of the refractive indices n_2/n_1 of the two media, or equivalently, to the ratio of the phase velocities v_1/v_2 in the two media:
>
> $$\frac{\sin\theta_1}{\sin\theta_2} = n_{2,1} = \frac{n_2}{n_1} = \frac{v_1}{v_2}.$$

5. (Chinese *Gaokao*) A man pulls a rope connected to a boat horizontally with constant speed v, as shown. Compute the tangential and normal components of the velocity and acceleration of the boat.

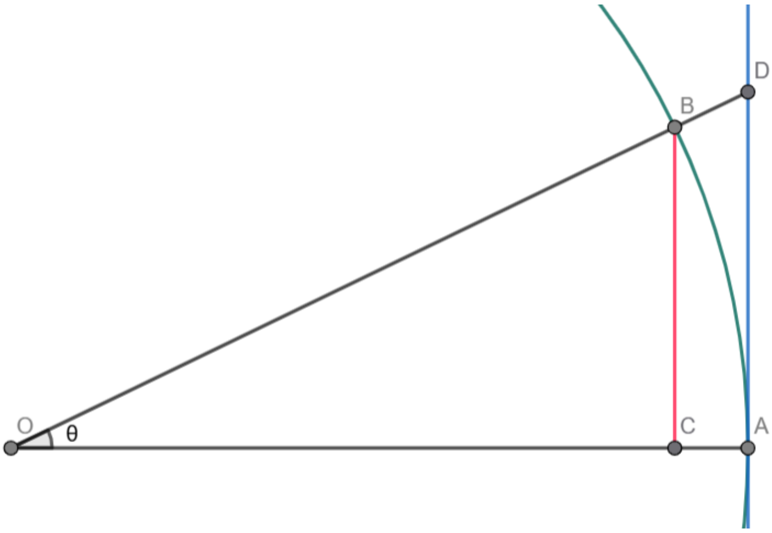


5 Solutions.

5.1 Problem 1.

As shown in the figure,

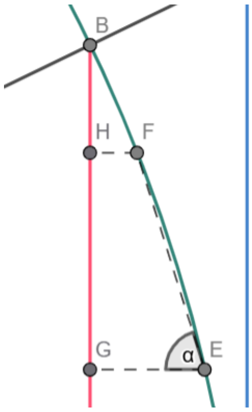
$$\sin\theta = |BC|\,, \quad \theta = \text{arclength of } AB, \quad \tan\theta = |AD|\,.$$



First, horizontally partition the segment BC and the arc AB into pieces, one of which is denoted by the pair consisting of segment GH and arc EF. When the partition is fine enough,

$$|GH| = |EF|\sin\alpha \approx (\text{arclength of } EF)\sin\alpha.$$

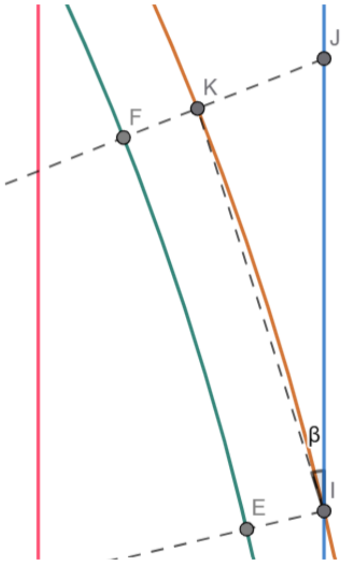
Adding up, we conclude that $\sin\theta < \theta$.



Next, radially partition the arc AB and the segment AD into pieces, one of which is denoted by the pair consisting of arc EF and segment IJ. Consider a concentric intermediate arc IK. When the partition is fine enough,

$$\text{arclength of } EF < \text{arclength of } IK \approx |IK| \approx |IJ|\cos\beta.$$

Adding up, we conclude that $\theta < \tan\theta$.



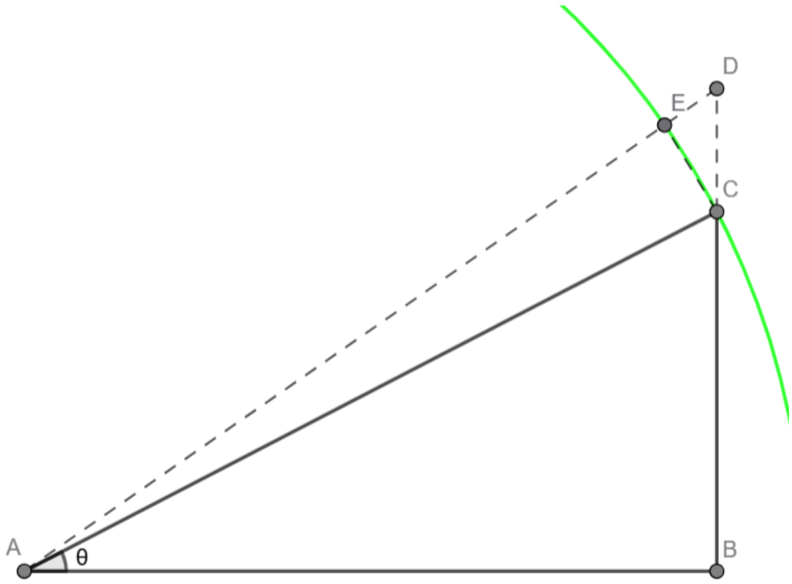
Now, as $\theta \to 0^+$, the angle α is uniformly close to $\pi/2$, so $|GH| \approx$ arclength of EF uniformly for all such partition pairs. Thus, $\sin\theta \approx \theta$. Similarly, β is uniformly close to 0, and the arc IK is infinitesimally close to the arc EF, so arclength of $EF \approx |IJ|$ uniformly for all such partition pairs. Hence, $\theta \approx \tan\theta$.

Remark. By taking advantage of these figures, potentially adding more auxiliary lines, one may prove that

$$(\sin\theta)' = \cos\theta, \quad (\tan\theta)' = \frac{1}{\cos^2\theta}.$$

Essentially, all familiar properties of trigonometric functions can be read off from these figures and from similar pictures that you may readily construct yourself.

5.2 Problem 2.



Construct a right triangle $\triangle ABC$ with $\angle B = \pi/2$ and

$$|AB| = 1, \quad |BC| = x.$$

Then $\arctan x = \theta = \angle BAC$. Let D be a vertical perturbation of C with $|CD| = \delta x$, and let E be the unique point on the segment AD such that $|AE| = |AC|$. We have

$$\delta\theta = \angle CAD = \frac{\text{arclength of } CE}{|AC|} \approx \frac{|CE|}{|BC| \,/\, \sin\theta} \approx \frac{|CD|\cos\theta}{|BC| \,/\, \sin\theta} = \frac{\delta x}{x}\cos\theta\sin\theta.$$

We conclude that the derivative of $\arctan x$ is

$$\frac{\delta\theta}{\delta x} = \frac{\cos\theta\sin\theta}{x} = \frac{1}{x} \cdot \frac{1}{\sqrt{x^2+1}} \cdot \frac{x}{\sqrt{x^2+1}} = \frac{1}{x^2+1}.$$

Remark. Imitate this approach to compute the derivatives of $\arcsin x$ and $\arccos x$. Additionally, prove that

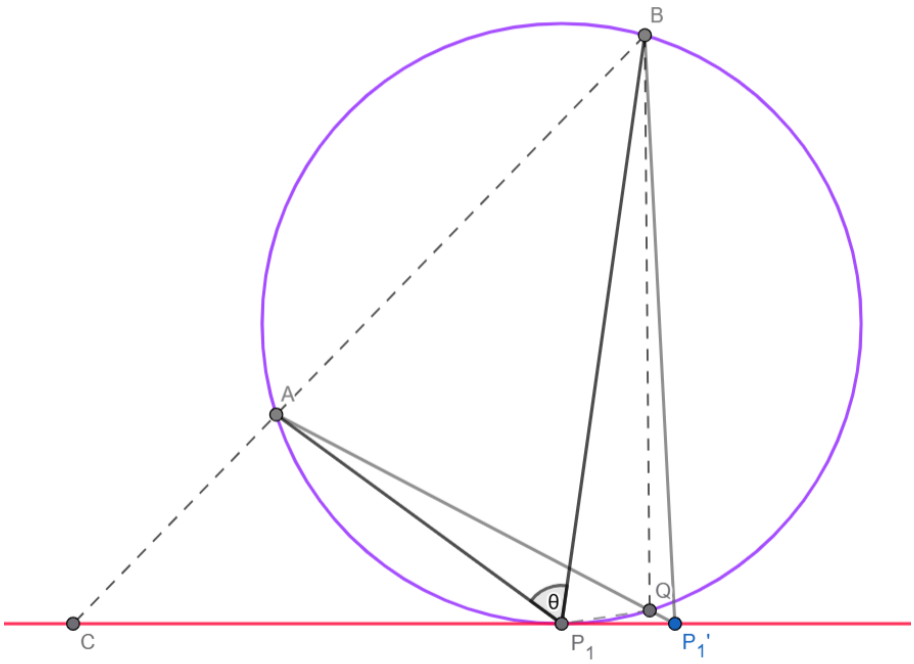
$$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta.$$

You may also begin to think about how to derive the relation between θ and x from the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\cos\theta\sin\theta}{x},$$

which will be covered in Math 125.

5.3 Problem 3.



We first consider the case where the line AB is not parallel to ℓ. Let P_1 be a point of local maximum. If we perturb P_1 to some point $P_1' \in \ell$ with $|P_1P_1'| = \delta x$, then

$$\delta\theta = |\angle AP_1'B - \angle AP_1B| \ll \delta x.$$

Let Q be the unique point on the ray AP_1' such that $\angle AQB = \theta$, and let R be the unique point on the ray BP_1' such that $|BQ| = |BR|$. Then

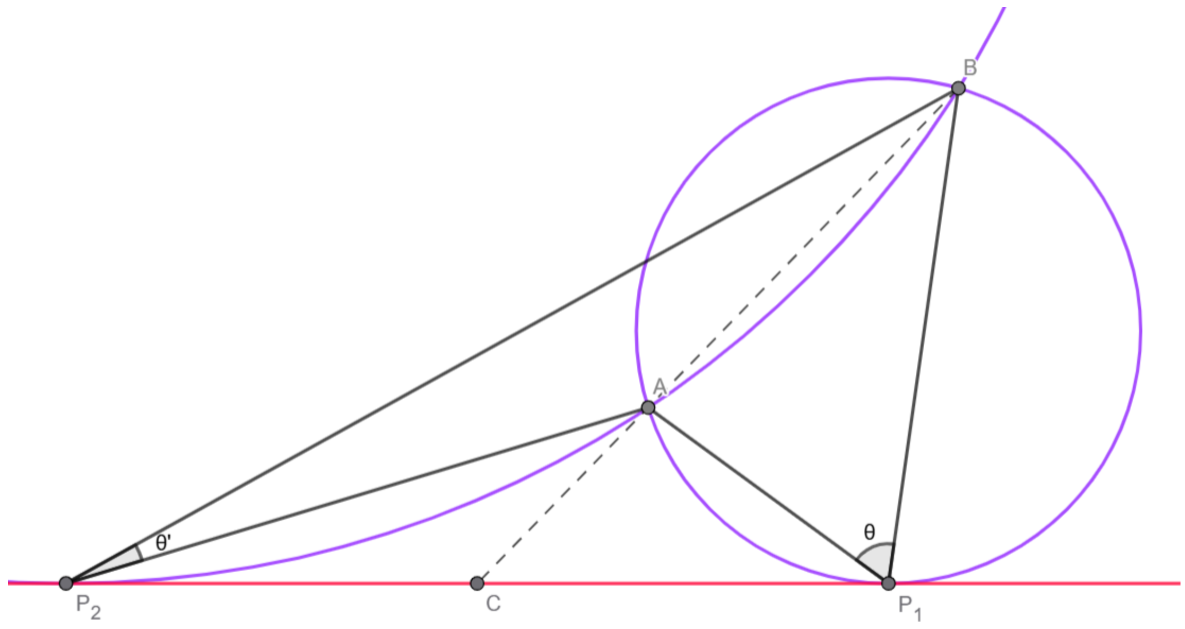
$$|QP_1'| \approx |QR| \sin\angle AP_1'B \approx |BQ| \,\angle QBP_1' \sin\theta \approx |BP_1'|\,\delta\theta\sin\theta \ll \delta x = |P_1P_1'|\,.$$

We conclude that

$$\angle CP_1A \approx \angle CP_1'A \approx \angle AQP_1' = \angle ABP_1,$$

where the last equality follows from the fact that A, B, Q, P_1 lie on the same circle. Hence, the circle ABP_1 is tangent to the line ℓ.

Since $|CA|\,|CB| = |CP_1|^2$, there must be another point $P_2 \in \ell$ on the other side of C satisfying the same equation. Thus, there are two points $P_1, P_2 \in \ell$ that locally maximize $\theta = \angle APB$, satisfying $|CA|\,|CB| = |CP_1|^2 = |CP_2|^2$.



If $AB \parallel \ell$, then one of the points P_i is pushed to infinity, so only one remains.

Remark. We have skipped some intermediate angle-transition arguments. For example, show that

$$\angle CP_1'A \approx \angle AQP_1'$$

using the fact that $|QP_1'| \ll |P_1P_1'|$ and the law of sines.

5.4 Problem 4.

By Fermat's principle of least time, light emitted from A and arriving at B will pass through some point $P \in \ell$ that minimizes the total travel time

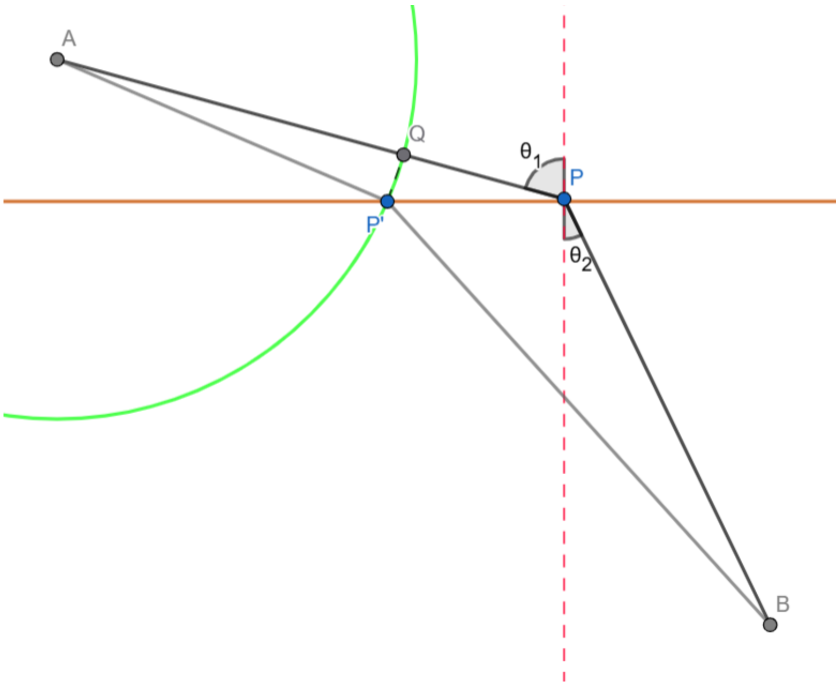
$$t = t_1 + t_2 = \frac{|AP|}{v_1} + \frac{|BP|}{v_2}.$$

We perturb P to a nearby point $P' \in \ell$ with $|PP'| = \delta x$. Then

$$\delta t = |\delta t_1 - \delta t_2| = \left| \frac{\delta\,|AP|}{v_1} - \frac{\delta\,|BP|}{v_2} \right| \ll \delta x.$$

Let Q be the unique point on the segment AP such that $|AQ| = |AP'|$. Then

$$\angle PP'Q \approx \theta_1.$$



We have

$$\delta t_1 = \frac{\delta\,|AP|}{v_1} = \frac{||AP| - |AP'||}{v_1} = \frac{|PQ|}{v_1} \approx \frac{|PP'|\sin\theta_1}{v_1} = \frac{\delta x \sin\theta_1}{v_1}.$$

Similarly, $\delta t_2 = \dfrac{\delta\,|BP|}{v_2} \approx \dfrac{\delta x \sin\theta_2}{v_2}$. Therefore, the condition $\delta t \ll \delta x$ forces

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \quad\Longrightarrow\quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

Remark. One might initially worry about the legitimacy of these approximations. The exact expression for $\delta\,|AP|$ is $\delta x \sin(\theta_1 \pm \delta\theta_1)$, where the explicit formula for $\delta\theta_1$ is extremely complicated. However, this does not affect the validity of the argument.

5.5 Problem 5.

No picture is needed this time; the argument is purely theoretical. Our solution is based on the following formula for the *centripetal acceleration* a_{cp} of uniform circular motion:

$$a_{\text{cp}} = \frac{v^2}{r} = \omega^2 r = \omega v,$$

where v is the *transverse velocity* and ω is the *angular velocity*. Prove this formula using infinitesimal analysis before reading on.

Let $\top$ and $\perp$ denote the tangential and normal projection operators with respect to the rope, and let $\mathbf{v}$ and $\mathbf{a}$ be the velocity and acceleration of the boat. Since the rope is taut, the speed along the rope is constant:

$$\mathbf{v}^\top = v.$$

The boat moves horizontally, so $\mathbf{v}$ is horizontal and its vertical component vanishes:

$$\mathbf{v}^\top \sin\theta - \mathbf{v}^\perp \cos\theta = 0 \quad\Longrightarrow\quad \mathbf{v}^\perp = v\tan\theta.$$

We now compute the acceleration $\mathbf{a}$. The tangential component $\mathbf{a}^{\top}$ arises solely from the rotation of $\mathbf{v}^{\perp}$, since $\mathbf{v}^{\top}$ is constant:

$$\mathbf{a}^{\top} = \frac{(\mathbf{v}^{\perp})^2}{r} = \frac{v^2 \tan^2\theta}{h/\sin\theta} = \frac{v^2}{h}\frac{\sin^3\theta}{\cos^2\theta}.$$

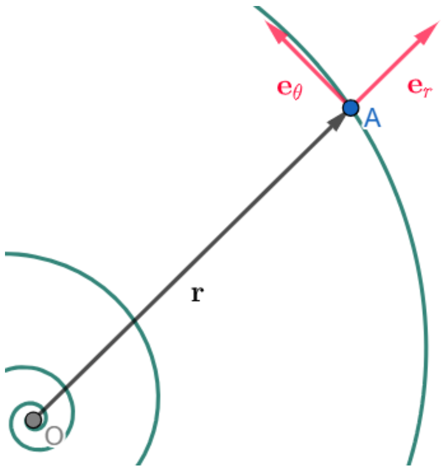
Since $\mathbf{a}$ is also horizontal, its vertical component vanishes:

$$\mathbf{a}^{\top}\sin\theta - \mathbf{a}^{\perp}\cos\theta = 0 \quad \Longrightarrow \quad \mathbf{a}^{\perp} = \mathbf{a}^{\top}\tan\theta = \frac{v^2}{h}\frac{\sin^4\theta}{\cos^3\theta}.$$

We may also compute the magnitudes of $\mathbf{v}$ and $\mathbf{a}$:

$$\begin{aligned}
|\mathbf{v}| &= \mathbf{v}^{\top}\cos\theta + \mathbf{v}^{\perp}\sin\theta = \frac{v}{\cos\theta}, \\
|\mathbf{a}| &= \mathbf{a}^{\top}\cos\theta + \mathbf{a}^{\perp}\sin\theta = \frac{v^2}{h}\tan^3\theta.
\end{aligned}$$

Remark. If this problem were to appear in Math 124, one would need to differentiate s_0 twice with respect to t, which is far from simple. The value of the solution presented here is that it avoids heavy computation and instead assigns clear physical interpretations to each quantity.



Finally, we derive the velocity and acceleration of a particle in polar coordinates. Let $\mathbf{r} = r\mathbf{e}_r$ be the position vector, and let $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ be the radial and transverse orthonormal frame, where $\mathbf{e}_r$ points outward and $\mathbf{e}_\theta$ points counterclockwise. Then

$$\mathbf{v}_r = r', \quad \mathbf{v}_\theta = \omega r = \theta' r, \quad \Longrightarrow \quad \mathbf{v} = r'\mathbf{e}_r + r\theta'\mathbf{e}_\theta.$$

Before differentiating, recall that unlike the stationary Cartesian frame $\{\mathbf{e}_1, \mathbf{e}_2\}$, the polar frame $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ is rotating. In particular,

$$\mathbf{e}_r' = \theta'\mathbf{e}_\theta, \quad \mathbf{e}_\theta' = -\theta'\mathbf{e}_r.$$

Using the product rule,

$$\begin{aligned}
\mathbf{a} = \mathbf{v}' &= r''\mathbf{e}_r + r'\theta'\mathbf{e}_\theta + (r'\theta' + r\theta'')\mathbf{e}_\theta - r(\theta')^2\mathbf{e}_r \\
&= (r'' - r(\theta')^2)\mathbf{e}_r + (2r'\theta' + r\theta'')\mathbf{e}_\theta.
\end{aligned}$$

The term $2r'\theta'$ is closely related to Coriolis-type effects in rotating coordinate systems.