The History of Infinity

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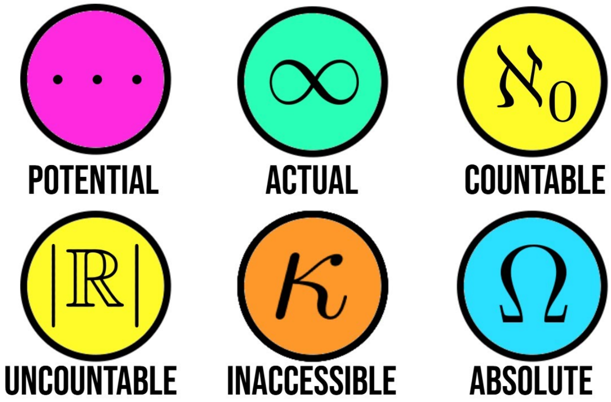
Introduction

Few concepts in mathematics have stirred as much debate as infinity. Unlike numbers we can count or quantities we can measure, infinity resists completion. It is both a process and a state, both intuitive and counterintuitive. From ancient Greek paradoxes to modern set theory, the discussion of infinitude has shaped mathematics and philosophy alike.



Zeno's Paradox

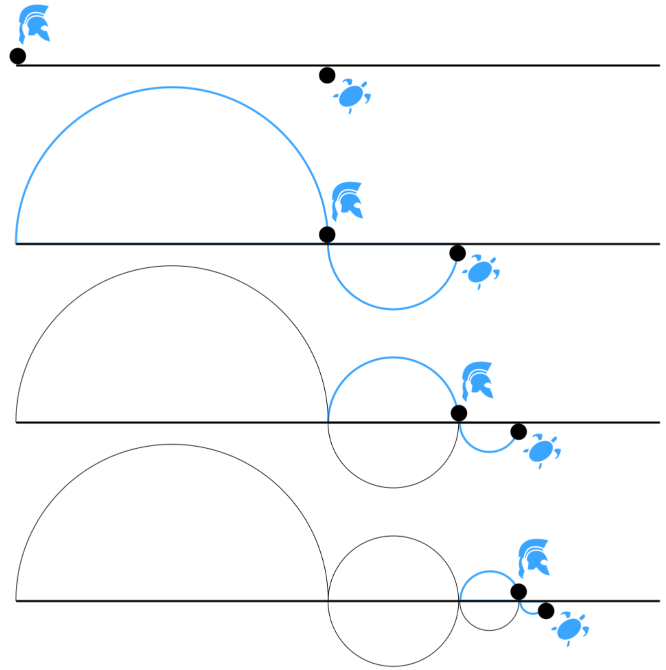
Zeno devised these paradoxes to support his teacher Parmenides's philosophy of **monism**, which posits that despite people's sensory experiences, *reality is singular and unchanging*. The paradoxes famously challenge the notions of **plurality** (the existence of many things), motion, space, and time by suggesting they lead to logical contradictions.

Zeno's work, primarily known from second-hand accounts since his original texts are lost, comprises forty "paradoxes of plurality", which argue against the coherence of believing in multiple existences, and several arguments against motion and change. Of these, only a few are definitively known today, including the renowned **"Achilles Paradox"**, which illustrates the problematic concept of *infinite divisibility in space and time*.

Zeno's Paradox

In the paradox of **Achilles and the tortoise**, Achilles is in a footrace with a tortoise. Achilles allows the tortoise a head start of 100 meters, for example. Suppose that each racer starts running at some constant speed, one faster than the other. After some finite time, Achilles will have run 100 meters, bringing him to the tortoise's starting point. During this time, the tortoise has run a much shorter distance, say 2 meters. It will then take Achilles some further time to run that distance, by which time the tortoise will have advanced farther; and then more time still to reach this third point, while the tortoise moves ahead. Thus, *whenever Achilles arrives somewhere the tortoise has been, he still has some distance to go before he can even reach the tortoise.*

Zeno's Paradox

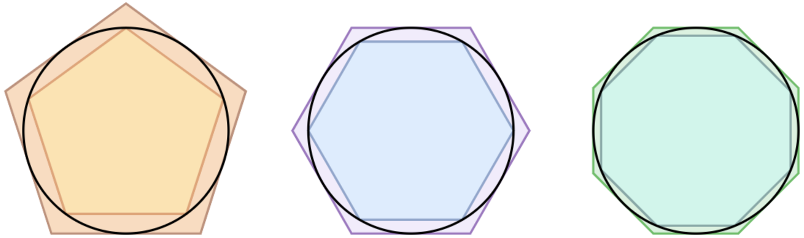


Aristotle's Distinction

Aristotle handled the topic of infinity in *Physics and in Metaphysics*. He distinguished between *actual and potential infinity*. **Actual infinity** is completed and definite, and consists of infinitely many elements. **Potential infinity** is never complete: elements can always be added, but never infinitely many. Aristotle distinguished between infinity with respect to **addition** and **division**. As an example of a potentially infinite series with respect to increase, one number can always be added after another in the series that starts $1, 2, 3, \ldots$, but the process of adding more and more numbers cannot be exhausted or completed. With respect to division, a potentially infinite sequence of divisions might start, for example, $1, 1/2, 1/4, 1/8, \ldots$, but the process of division cannot be exhausted or completed.

Method of Exhaustion

It is a method of finding the area of a shape by *inscribing inside it a sequence of polygons* (one at a time) whose areas converge to the area of the containing shape. If the sequence is correctly constructed, the difference in area between the n-th polygon and the containing shape will become arbitrarily small as n becomes large. As this difference becomes arbitrarily small, the possible values for the area of the shape are systematically "exhausted" by the lower bound areas successively established by the sequence members. (Euclid, Archimedes)



Theological Debates

Finitistic theism denies that God is omnipotent. Ray Harbaugh Dotterer, in his book *The Argument for a Finitist Theology* (1917), summarized the argument for theistic finitism:

> *God can not be thought to be at once* **omnipotent** *and* **perfectly good**. *If we say that he is omnipotent, that his sovereignty is complete, that all events that occur are willed by him; then it follows that he is responsible for the actual world, which is partly evil, and, accordingly, that he is not perfectly good. If we begin at the other end, and say God is perfectly good, then we must deny that he is omnipotent.*

The idea that *God is and must be infinite* has been a nearly universal belief amongst monotheists. Only a minority of thinkers have advanced the idea of a finite deity.

Descartes and Pascal's Viewpoint

- Descartes acknowledged infinity as a necessary concept, particularly in **geometry**, but viewed it as *an attribute belonging solely to God*. He argued that humans could not fully comprehend infinity and instead used the term "indefinite" for things like the extension of the universe or the divisibility of matter, to distinguish human understanding from divine omniscience.
- Pascal wrestled profoundly with the concept in his work, most famously in his *Pensées*. He explored *the human position between the infinitely large and the infinitely small*, which he saw as evidence of **human wretchedness and the grandeur of God**.

Galileo's Paradox

In his final scientific work, *Two New Sciences*, Galileo Galilei made apparently contradictory statements about the **positive integers**. First, a square is an integer that is the square of an integer. Some numbers are squares, while others are not; therefore, all the numbers, including both squares and non-squares, must be more numerous than just the squares. And yet, for every number there is exactly one square; hence, there cannot be more of one than of the other. This is an early use, though not the first, of the idea of **one-to-one correspondence in the context of infinite sets**. Galileo concluded that the ideas of *less*, *equal*, and *greater* apply to finite quantities but not to infinite quantities.

Calculus and Infinitesimals

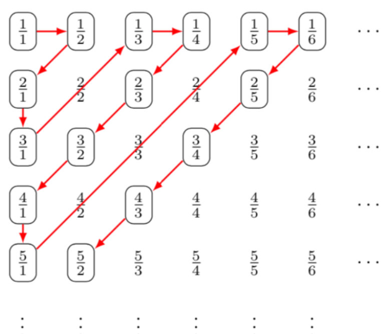
- **Newton** and **Leibniz** developed calculus using *infinitesimals quantities* smaller than any finite number but not zero. This was a bold embrace of infinity, though critics found it logically shaky.
- *Limits* later replaced infinitesimals in mainstream analysis, formalized by **Cauchy** and **Weierstrass** in the 19th century. Limits allowed mathematicians to rigorously define *infinite sums, series, and convergence*.
- Still, infinitesimals returned in the 20th century through **nonstandard analysis**, showing that infinity could be handled in multiple consistent frameworks.

Cantor and Set Theory

Cantor proved that *some infinities are larger than others*.

- The set of natural numbers is **countably infinite**.
- The set of real numbers is **uncountably infinite**, a strictly larger infinity

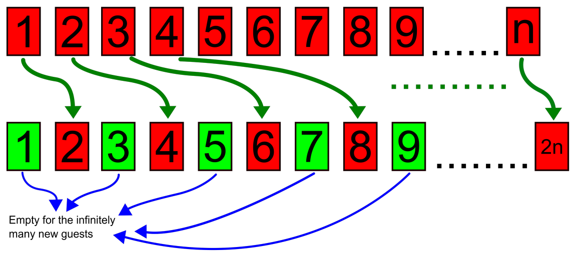
He introduced the concept of **cardinality** to compare infinite sets, and the famous **diagonal argument** to show that real numbers cannot be listed in sequence.



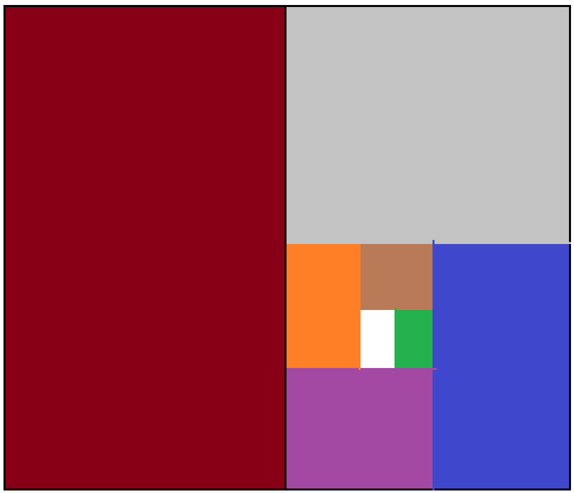
Paradoxes and Foundations

Russell's paradox revealed contradictions in naive set theory: *the set of all sets that do not contain themselves both must and must not contain itself*. This led to the development of **axiomatic set theory (Zermelo–Fraenkel with Choice, ZFC)**, which remains the foundation of modern mathematics.

Hilbert's Hotel, a thought experiment, dramatized the counterintuitive properties of infinity: *a fully occupied infinite hotel can still accommodate new guests by shifting everyone one room over*.



Analysis and Geometry



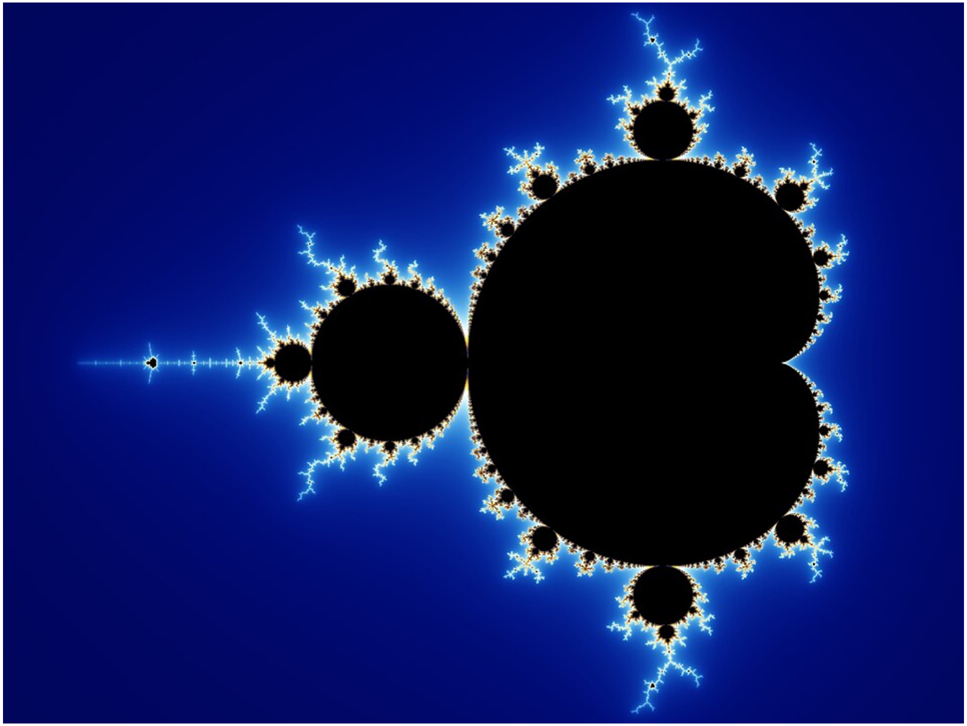
The geometric ratio is 1/2...

The first term is 1/2

$$S_\infty = \frac{1/2}{1 - 1/2} = 1$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots = 1$$

Analysis and Geometry



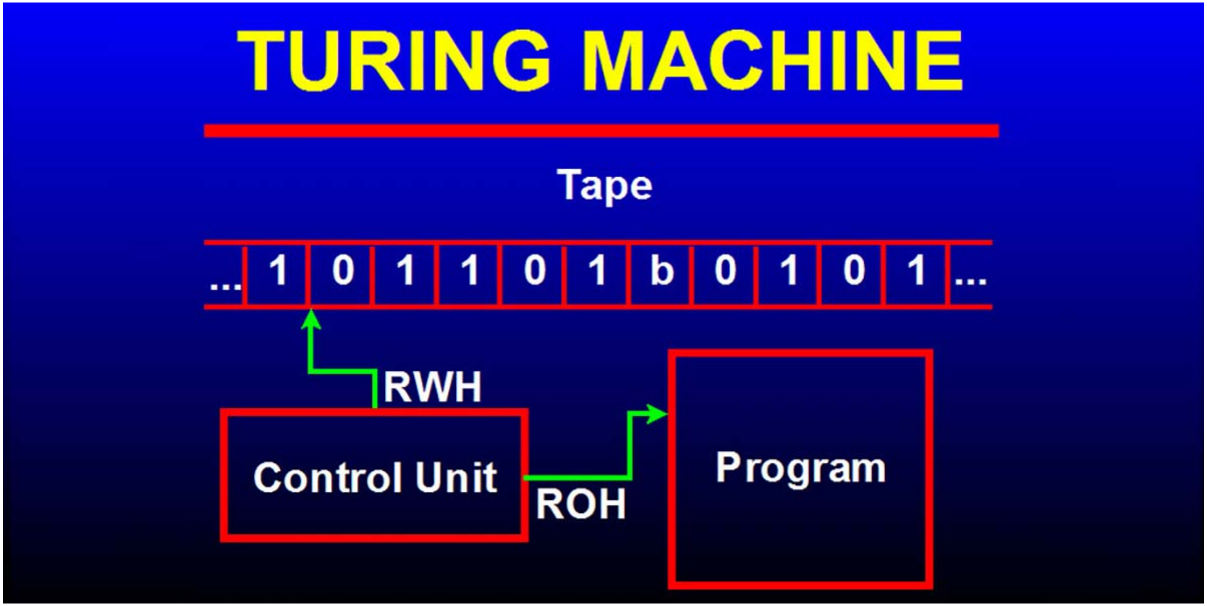
Logic and Computation

Gödel's first incompleteness theorem states that no consistent system of axioms whose theorems can be listed by an effective procedure (i.e., an algorithm) is capable of proving all truths about the arithmetic of natural numbers. For any such consistent formal system, there will always be statements about natural numbers that are *true but that are unprovable within the system*. Equivalently, there will always be statements about natural numbers that are false, but that are unprovably false within the system.
Gödel's second incompleteness theorem, an extension of the first, shows that the system cannot demonstrate its own consistency.

Logic and Computation

A **Turing machine** operates on an **infinite** *memory tape* divided into *discrete cells*, each of which can hold a single symbol drawn from a finite set of symbols called the *alphabet of the machine*. It has a "head" that, at any point in the machine's operation, is positioned over one of these cells, and a "state" selected from a finite set of states. At each step of its operation, the head reads the symbol in its cell. Then, based on the symbol and the machine's own present state, the machine writes a symbol into the same cell, and moves the head one step to the left or the right, or halts the computation. The choice of which replacement symbol to write, which direction to move the head, and whether to halt is based on a *finite table* that specifies what to do for each combination of the current state and the symbol that is read.

Logic and Computation



Conclusion

- Infinity is not just a mathematical curiosity; it is a cornerstone of modern thought. From Zeno's paradoxes to Cantor's set theory, from calculus to logic, infinity forces us to confront the limits of human intuition. Mathematicians have tamed it with rigorous definitions, but the quarrels remain alive in public discourse.
- Infinity is both a tool and a mystery: indispensable for mathematics, yet endlessly provocative. The discussion of infinitude will never end—appropriately enough, for a concept that itself has no end.