Rotations on the Riemann Sphere

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Problem. We aim to find the equivalent condition for the Möbius transform on the Riemann sphere to be a rotation.

Let $\pi : \mathbb{C}_\infty \to \mathbb{S}^2 \subset \mathbb{R}^3$ be the (inverse) stereographic projection with

$$\pi(z) = \begin{cases} \left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right), & z \neq \infty, \\ (\mathbf{0}, 1), & z = \infty. \end{cases}$$

For any matrix $A \in \mathrm{GL}_2(\mathbb{C})$, the Möbius transform $\mu_A : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is uniquely determined by the commutative diagram

$$\begin{tikzcd} \mathbb{C}^2 \setminus \{0\} \arrow[r, "A", "\cong"'] \arrow[d, two heads, "q"'] & \mathbb{C}^2 \setminus \{0\} \arrow[d, two heads, "q"] \\ \mathbb{C}_\infty \arrow[r, dashed, "\mu_A", "\cong"'] & \mathbb{C}_\infty \end{tikzcd}$$

where $q : \mathbb{C}^2 \setminus \{0\} \twoheadrightarrow \mathbb{C}_\infty$ is defined by

$$q(z_1, z_2) = \begin{cases} \frac{z_1}{z_2}, & z_2 \neq 0, \\ \infty, & z_2 = 0. \end{cases}$$

Combined with the stereographic projection, we have the commutative diagram

$$\begin{tikzcd} \mathbb{C}^2 \setminus \{0\} \arrow[r, "A", "\cong"'] \arrow[d, two heads, "q"'] & \mathbb{C}^2 \setminus \{0\} \arrow[d, two heads, "q"] \\ \mathbb{C}_\infty \arrow[r, "\mu_A", "\cong"'] \arrow[d, "\pi"', "\cong"] & \mathbb{C}_\infty \arrow[d, "\pi", "\cong"'] \\ \mathbb{S}^2 \arrow[r, dashed, "\widetilde{\mu_A}", "\cong"'] & \mathbb{S}^2 \end{tikzcd}$$

where $\widetilde{\mu_A} : \mathbb{S}^2 \to \mathbb{S}^2$ is uniquely determined by μ_A as an automorphism of $\mathbb{S}^2$.

We immediately compute, for any $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$,

$$\pi \circ q(\mathbf{z}) = \left(\frac{2z_1\overline{z_2}}{|z_1|^2 + |z_2|^2}, \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2} \right),$$

which is true even if $z_2 = 0$. Thus for any $\mathbf{z}, \mathbf{w} \in \mathbb{C}^2 \setminus \{0\}$,

$$\langle \pi \circ q(\mathbf{z}), \pi \circ q(\mathbf{w}) \rangle_{\mathbb{R}^3} = \frac{2(z_1\overline{z_2}\,\overline{w_1}w_2 + \overline{z_1}z_2w_1\overline{w_2}) + (|z_1|^2 - |z_2|^2)(|w_1|^2 - |w_2|^2)}{(|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2)},$$

where we have used the connection between Euclidean and Hermitian inner products:

$$\langle z, w \rangle_{\mathbb{R}^2} = \frac{1}{2}(z \cdot w + w \cdot z).$$

The Lagrange identity states that for $\mathbf{z}, \mathbf{w} \in \mathbb{C}^2$, $|\mathbf{z} \cdot \mathbf{w}|^2 + |\mathbf{z} \times \mathbf{w}|^2 = \|\mathbf{z}\|^2 \|\mathbf{w}\|^2$, namely,

$$|z_1\overline{w_1} + z_2\overline{w_2}|^2 + |z_1w_2 - z_2w_1|^2 = (|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2).$$

We deduce that the numerator

$$\begin{aligned}
&2(z_1\overline{z_2}\,\overline{w_1}w_2+\overline{z_1}z_2w_1\overline{w_2})+(|z_1|^2-|z_2|^2)(|w_1|^2-|w_2|^2)\\
&\qquad=(|z_1|^2+|z_2|^2)(|w_1|^2+|w_2|^2)-2(|z_1|^2|w_2|^2+|z_2|^2|w_1|^2)+2(z_1\overline{z_2}\,\overline{w_1}w_2+\overline{z_1}z_2w_1\overline{w_2})\\
&\qquad=(|z_1|^2+|z_2|^2)(|w_1|^2+|w_2|^2)-2\,|z_1w_2-z_2w_1|^2\\
&\qquad=2\,|z_1\overline{w_1}+z_2\overline{w_2}|^2-(|z_1|^2+|z_2|^2)(|w_1|^2+|w_2|^2),
\end{aligned}$$

and consequently,

$$\langle\pi\circ q(\mathbf{z}),\pi\circ q(\mathbf{w})\rangle_{\mathbb{R}^3}=2\frac{|z_1\overline{w_1}+z_2\overline{w_2}|^2}{(|z_1|^2+|z_2|^2)(|w_1|^2+|w_2|^2)}-1=2\frac{|\mathbf{z}\cdot\mathbf{w}|^2}{\|\mathbf{z}\|^2\,\|\mathbf{w}\|^2}-1.$$

By writing $\widehat{\mathbf{z}}=\mathbf{z}/\|\mathbf{z}\|,\widehat{\mathbf{w}}=\mathbf{w}/\|\mathbf{w}\|$, this can be further simplified to

$$\langle\pi\circ q(\mathbf{z}),\pi\circ q(\mathbf{w})\rangle_{\mathbb{R}^3}=2\,|\widehat{\mathbf{z}}\cdot\widehat{\mathbf{w}}|^2-1.$$

We could have considered merely $\pi\circ q|_{\mathbb{S}^3}$ for the unit sphere $\mathbb{S}^3\subset\mathbb{C}^2\setminus\{0\}$, so that the hats can be removed from $\widehat{\mathbf{z}},\widehat{\mathbf{w}}$.

Indeed, $\widetilde{\mu_A}$ must be orientation-preserving. On one hand, $\widetilde{\mu_A}=\pi\circ\mu_A\circ\pi^{-1}$ shares the same orientation with μ_A. On the other hand, since $\mu_A=u+iv$ is biholomorphic, by the Cauchy-Riemann equation,

$$\det\mathrm{d}\mu_A=\det\begin{pmatrix}\partial_x u & \partial_y u\\ \partial_x v & \partial_y v\end{pmatrix}=(\partial_x u)^2+(\partial_y u)^2>0.$$

Then μ_A is orientation-preserving, and so does $\widetilde{\mu_A}$. Additionally, it is well-known that $\mathrm{O}(n)\cong\mathrm{SO}(n)\rtimes\mathbb{Z}/2$, where $\mathrm{SO}(n)\lhd\mathrm{O}(n)$ is the special orthogonal group of all rotations of $\mathbb{S}^2$ when $n=3$.

So far, for any $A\in\mathrm{GL}_2(\mathbb{C})$, the following facts are true:

$$\begin{aligned}
\langle\pi\circ q(\mathbf{z}),\pi\circ q(\mathbf{w})\rangle_{\mathbb{R}^3}&=2\,|\widehat{\mathbf{z}}\cdot\widehat{\mathbf{w}}|^2-1,\\
\langle\pi\circ q\circ A(\mathbf{z}),\pi\circ q\circ A(\mathbf{w})\rangle_{\mathbb{R}^3}&=2\left|\widehat{A\mathbf{z}}\cdot\widehat{A\mathbf{w}}\right|^2-1,\\
\langle\pi\circ q\circ A(\mathbf{z}),\pi\circ q\circ A(\mathbf{w})\rangle_{\mathbb{R}^3}&=\langle\widetilde{\mu_A}\circ\pi\circ q(\mathbf{z}),\widetilde{\mu_A}\circ\pi\circ q(\mathbf{w})\rangle_{\mathbb{R}^3}.
\end{aligned}$$

Since $\pi\circ q$ is surjective,

$$\begin{aligned}
\widetilde{\mu_A}\text{ is a rotation}&\iff\widetilde{\mu_A}\text{ is orthogonal}\\
&\iff\langle\widetilde{\mu_A}\circ\pi\circ q(\mathbf{z}),\widetilde{\mu_A}\circ\pi\circ q(\mathbf{w})\rangle_{\mathbb{R}^3}=\langle\pi\circ q(\mathbf{z}),\pi\circ q(\mathbf{w})\rangle_{\mathbb{R}^3},\quad\forall\,\mathbf{z},\mathbf{w}\\
&\iff\left|\widehat{A\mathbf{z}}\cdot\widehat{A\mathbf{w}}\right|=|\widehat{\mathbf{z}}\cdot\widehat{\mathbf{w}}|,\quad\forall\,\mathbf{z},\mathbf{w}.
\end{aligned}$$

We are to show this is true if and only if A is conformal.

It remains a linear algebra problem to characterize all such A. Observe that the Hermitian matrix $B=A^\dagger A$ has a unitary diagonalization $U^\dagger DU$ for a unitary U and diagonal $D=\operatorname{diag}\{\lambda_1,\lambda_2\}$. Writing $\mathbf{z}'=U\mathbf{z},\mathbf{w}'=U\mathbf{w}$, and $A'=AU^{-1}$, we have $(A')^\dagger A'=D$ and

$$\left|\widehat{A\mathbf{z}}\cdot\widehat{A\mathbf{w}}\right|=|\widehat{\mathbf{z}}\cdot\widehat{\mathbf{w}}|\iff\left|\widehat{A'\mathbf{z}'}\cdot\widehat{A'\mathbf{w}'}\right|=\left|\widehat{\mathbf{z}'}\cdot\widehat{\mathbf{w}'}\right|\iff\frac{\left|(\mathbf{z}')^\dagger D\mathbf{w}'\right|}{\|A'\mathbf{z}'\|\,\|A'\mathbf{w}'\|}=\frac{\left|(\mathbf{z}')^\dagger\mathbf{w}'\right|}{\|\mathbf{z}'\|\,\|\mathbf{w}'\|}.$$

Taking $\mathbf{z}'=\mathbf{e}_1,\mathbf{w}'=\mathbf{e}_2$, we have

$$\mathbf{e}_1^\dagger D\mathbf{e}_2=0=\mathbf{e}_2^\dagger D\mathbf{e}_1.$$

Taking $\mathbf{z}'=\mathbf{e}_1+\mathbf{e}_2,\mathbf{w}'=\mathbf{e}_1-\mathbf{e}_2$, we have

$$(\mathbf{e}_1+\mathbf{e}_2)^\dagger D(\mathbf{e}_1-\mathbf{e}_2)=\mathbf{e}_1^\dagger D\mathbf{e}_1-\mathbf{e}_2^\dagger D\mathbf{e}_2=\lambda_1-\lambda_2=0,$$

so D is scalar and A is a scalar multiple of a unitary matrix. Conversely, if A is a scalar multiple of a unitary matrix, then $\left|\widehat{A\mathbf{z}}\cdot\widehat{A\mathbf{w}}\right|=|\widehat{\mathbf{z}}\cdot\widehat{\mathbf{w}}|$ is true by the definition of unitary matrices.

In summary, the automorphism of the sphere induced by an invertible 2×2 complex matrix in terms of Möbius transform is a rotation if and only if the matrix is a scalar multiple of a unitary matrix:

$$\mathrm{Rot}(\mathbb{S}^2)\cong\mathrm{PSU}(2).$$