Radon Measures &
Riesz Representation Theorems

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I am much obliged to my teacher, Professor Dongsheng Li, for inviting me to this high-quality seminar on geometric measure theory. I enjoy the dynamic and profound discussions shared and learned. Everyone will benefit much from these meaningful weekly conventions.

1 Universal Notations

Unless otherwise stated, X will denote a *locally compact Hausdorff (LCH) space*, U, V, W, O denote open sets, and K denotes compact sets. Unqualified functions should be continuous.

2 Topological Results

Definition 1.1 (Hausdorffness.) A topological space X is said to be **Hausdorff** if for any distinct $x, y \in X$, there exists $U \ni x$, $V \ni y$, such that $U \cap V = \varnothing$.

Remark:

- **All compact sets in X are closed.** Fix $x \notin K$. For each $y \in K$, there exists disjoint $U_y \ni x$ and $V_y \ni y$. $\{V_y\}_{y \in K}$ covers K, which has a subcover $\{V_{y_i}\}_{i=1}^N$. Now, $\bigcap_{i=1}^N U_{y_i}$ is an open set containing x and disjoint from K. Since x is arbitrary, $K^{\complement}$ is open.
- We can prove a bit more. Notice that we have separated a compact set and a point with open sets in X. This can be done analogously for *finitely many compact sets* as well. This process is iterative because finite intersections of open sets are open.
- **A compact Hausdorff space is normal**. This follows from the fact that closed subsets of compact sets are compact in general topological spaces.

Definition 1.2 (Local Compactness.) A topological space X is said to be **locally compact** if for each $x \in X$, there exists $U \ni x$ whose closure is compact. (Such neighborhoods are said to be *precompact*.)

Combined with Hausdorffness, we have the following crucial technical lemma:

Lemma 1.1 (Localization of Precompact Neighborhoods.) If X is LCH, and V is a neighborhood of some $x \in X$. Then there is a neighborhood $U_x \subset\subset V$. This means that $x \in U_x \subset \overline{U_x} \subset V$, and $\overline{U_x}$ is compact.

Proof: Let W_0 be a precompact neighborhood of x and let $W = W_0 \cap V$. Then $\overline{W}$ is normal as a subspace of X. Since $\partial W = \overline{W} \setminus W$ and $\{x\}$ are disjoint and closed, there exists U_1, U_2 of X such that

$$x \in U_1, \quad \partial W \subset U_2, \quad U_1 \cap U_2 \cap \overline{W} = \varnothing.$$

The lemma is proved with $U_x = U_1 \cap W \subset\subset \overline{W} \setminus U_2 \subset W \subset V$. $\square$

Remark:

- We can optimize the separators of finitely many compact sets to have disjoint closures.
- The single point in the lemma can naturally replaced by a compact set.

Lemma 1.2 (Urysohn's Lemma.) Let $K \subset V \subset X$. Then there exists f with $K \prec f \prec V$, which means there exists W, U such that

$$K \subset W \subset U \subset\subset V, \quad \mathbf{1}_W \leq f \leq \mathbf{1}_U.$$

Proof. By Lemma 1.1, for each $x \in K$, there exists a neighborhood $U_x \subset\subset V$. Then K is covered by $U = \bigcup_{j=1}^N U_{x_j} \subset\subset V$ for some finite collection $\{x_j\} \subset K$. Now $\overline{U}$ is compact, thus normal, so the *traditional Urysohn's lemma* gives an f_0 on $\overline{U}$ with

$$f_0|_K = 1, \quad f_0|_{\partial U} = 0,$$

which can be trivially extended to all of X by the *pasting lemma*. By lifting $f_0|_{\left\{f_0 > \frac{1}{2}\right\}}$ to 1, we let $f_1 = \min\{2f_0, 1\}$. This is the desired construction. $\square$

Corollary 1.3 (Partition of Unity.) If $K \subset X$ is covered by $\{U_j\}_{j=1}^N$, there exists functions $\{\varphi_j\}_{j=1}^N$ such that

$$\varphi_j \prec U_j, \quad \sum_j \varphi_j \succ K.$$

Proof. We first refine the covering to a precompact covering. For any $x \in K$ there is a neighborhood U_x and $j = j(x)$ such that $U_x \subset\subset U_j$, and suppose K is covered by $\{U_{x_i}\}_{i=1}^{N_0}$. Now set

$$V_j = \bigcup_{U_{x_i} \subset U_j} U_{x_i} \subset\subset U_j$$

to assemble the refinement. By Lemma 1.2, we can select functions $\{\psi_j\}$ and f_0 such that

$$\overline{V_j} \prec \psi_j \prec U_j, \quad \bigcup_j \overline{V_j} \prec f_0 \prec \mathbf{1}_{\left\{\sum_j \psi_j > 0\right\}}.$$

f_0 is introduced so that $\sum_{i=0}^N \psi_i > 0$ everywhere for $\psi_0 = 1 - f_0$. Finally, noting that $\sum_{i\geq 1} \psi_i = \sum_{i \geq 0} \psi_i$ on $V = \bigcup_j V_j \supset K$, the normalizations of ψ_j,

$$\varphi_j = \frac{\psi_j}{\sum_{i\geq 1} \psi_i}, \quad 1 \leq j \leq N,$$

satisfies the requirements. $\square$

3 Radon Measures

Though the definition and properties we introduce are also valid in Hausdorff spaces, Radon measures are most commonly used in locally compact Hausdorff spaces.

Definition 2.1 (Radon Measures.) A **Radon measure** on X is an outer measure μ satisfying:

- R1: μ is Borel regular and μ is finite on all compact sets.
- R2: μ is outer regular on all subsets in the sense that $\mu(A) = \inf_{U \supset A} \mu(U)$.
- R3: μ is inner regular on all *open* sets in the sense that $\mu(U) = \sup_{K \subset U} \mu(K)$.

Lemma 2.1 (A Derivation from R3.) If μ is Radon, then for any μ-measurable set A with finite measure,

$$\mu(A) = \sup_{K \subset A} \mu(K).$$

Proof. Since μ is a measure on all μ-measurable subsets, and $A, K, U \setminus A$ are all μ-measurable by the Borel regularity, we can choose $U \supset A$ with $\mu(U \setminus A) < \varepsilon$, $K \subset U$ with $\mu(U \setminus K) < \varepsilon$, and $W \supset U \setminus A$ with $\mu(W \setminus (U \setminus A)) < \varepsilon$, as by definition

$$\mu(U) = \mu(A) + \mu(U \setminus A) = \mu(K) + \mu(U \setminus K), \quad \mu(W) = \mu(W \setminus (U \setminus A)) + \mu(U \setminus A).$$

Then we can approximate A from below by the compact set $K \setminus W$:

$$\mu(K \setminus W) \geq \mu(K) - \mu(W) = \mu(U) - \mu(U \setminus K) - \mu(W \setminus (U \setminus A)) - \mu(U \setminus A) \geq \mu(A) - 3\varepsilon.$$

$\square$

The following lemma asserts that the property R1 follows from R2 and R3 provided that μ is finite and finitely additive on compact sets:

Lemma 2.2 (A Condition for R1) Assume that the outer measure μ satisfies R2 and R3, and in addition,

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2) < \infty$$

whenever K_1, K_2 are disjoint. Then R1 holds and thus μ is a Radon measure.

Proof. We need to prove that μ is Borel regular, that is to say,

- All Borel sets are μ-measurable,
- $\forall\, A \subset X$, $\exists$ a Borel set $B \supset A$, s.t. $\mu(B) = \mu(A)$.

R2 gives a sequence of open sets $\{U_j\}$ with $\mu(U_j) \leq \mu(A) + \frac{1}{j}$, so B can take $\bigcap_j U_j$. It suffices to show that open sets, the generators of Borel sets, are μ-measurable. That is,

$$\mu(Y \setminus U) + \mu(Y \cap U) \leq \mu(Y)$$

for any $\mu(Y) < \infty$. We ε-approximate Y with $V \supset Y$, $V \cap U$ with $K_1 \subset V \cap U$, and $V \setminus K_1$ with $K_2 \subset V \setminus K_1$, then $K_1 \cap K_2 = \varnothing$ and

$$\begin{aligned}\mu(Y \setminus U) + \mu(Y \cap U) &\leq \mu(V \setminus U) + \mu(V \cap U)\\ &\leq \mu(V \setminus K_1) + \mu(K_1) + \varepsilon\\ &\leq \mu(K_2) + \varepsilon + \mu(K_1) + \varepsilon\\ &= \mu(K_1 \cup K_2) + 2\varepsilon\\ &\leq \mu(V) + 2\varepsilon\\ &\leq \mu(Y) + 3\varepsilon.\end{aligned}$$

□

The following density result holds for the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$. We generalize this to Radon measures in LCH spaces:

Lemma 2.3 (A Dense Subspace of L^p.) For $1 \leq p < \infty$, $C_c(X)$ is dense in $L^p(X, \mu)$, for any Radon measure μ.

Proof. For any $f \in L^p$, both f^+ and f^- are non-negative measurable functions. The *Monotone Convergence Theorem* gives sequences of simple functions to approximate such functions monotonically. But simple functions are linear combinations of characteristic functions, so our goal is to approximate $\varphi = \mathbf{1}_A$ for A being μ-measurable and $\mu(A) < \infty$.

We ε-approximate A with $K \subset A$ and K with $U \supset K$. By Lemma 1.2, there exists $K \prec g \prec U$. Then $\varphi - g$ is supported on $(U \cup A) \setminus K = (U \setminus K) \cup (A \setminus K)$, a set of measure bounded by 2ε. Consequently, $\|\varphi - g\|_p \leq 2\varepsilon \cdot 1^p = 2\varepsilon$. □

4 Riesz Representation Theorems

We use $\mathcal{K}$ to denote $C_c(X; \mathbb{R})$ of all compactly supported continuous functions on X and $\mathcal{K}_+$ to denote non-negative functions in this set.

The first main result we will show implies the significance of Radon measures: to represent non-negative *locally bounded* linear functionals on $\mathcal{K}$ in terms of integral.

Theorem 3.1 (Riesz Representation Theorem No.1.) Let $\lambda : \mathcal{K}_+ \to \mathbb{R}_{\geq 0}$ be linear and locally bounded in the sense that

$$\sup_{f \prec K} \lambda(f) < \infty, \quad \forall K \subset X.$$

Then λ can be expressed as

$$\lambda(f) = \int_X f \, \mathrm{d}\mu$$

for a unique Radon measure μ and any $f \in \mathcal{K}_+$.

We will only show the existence part. The reader should remind that two Radon measures are the same if they overlap on all open sets, which is true if

$$\int_X f \, \mathrm{d}\mu_1 = \int_X f \, \mathrm{d}\mu_2, \quad \forall f \in \mathcal{K}_+.$$

Remark:

- Linear functionals on $\mathcal{K}$ can be uniquely determined by such λ, as we can set
$$\lambda(f) = \lambda\left(f^+\right) - \lambda\left(f^-\right), \quad \forall f \in \mathcal{K}.$$
- Observe that λ is **monotonic** in the sense that $f \geq g \Longrightarrow \lambda(f) \geq \lambda(g)$.
- If $f \prec K \prec g$, then $f \leq g \Longrightarrow \lambda(f) \leq \lambda(g)$.

It is immediate from the remark that

Lemma 3.1 (Another Monotonicity of λ.) For any $K \subset U \subset X$, we can choose $K \subset W \subset\subset U$ by Lemma 1.2. Then
$$\sup_{f \prec W} \lambda(f) \leq \inf_{g \succ \overline{W}} \lambda(g).$$

It follows that **the local boundedness requirement is redundant** out of monotonicity!

Lemma 3.2 (A Bound for λ.) Let μ be defined as in the proof of Theorem 3.1, then
$$\lambda(h) \leq (\sup h)\, \mu(x : h > 0).$$

Proof. This is true because h is the uniform increasing limit of $h_n = \left(h - \frac{1}{n}\right)^+$ in X as $n \to \infty$. Since λ is locally bounded,
$$\lambda(h_n) \nearrow \lambda(h), \quad \mu(\operatorname{supp} h_n) \nearrow \mu(x : h > 0), \quad \sup h_n \nearrow \sup h.$$
The details are totally embedded in the following proof. □

Proof of Theorem 3.1. We define
$$\mu(U) = \sup_{f \prec U} \lambda(f), \quad \mu(A) = \inf_{U \supset A} \mu(U), \quad \forall A \subset X.$$
Observe that in particular, $\mu(K)$ has the following expression:
$$\mu(K) = \inf_{g \succ K} \lambda(g) < \infty.$$
On one hand, for any $U \supset K$, there exists $K \prec h \prec U$, so that
$$\inf_{g \succ K} \lambda(g) \leq \lambda(h) \leq \sup_{g \prec U} \lambda(g) = \mu(U);$$
On the other hand, for each $g \succ K$, there exists $K \subset W \subset\subset X$ such that $g \succ \overline{W}$, so by Lemma 3.1,
$$\mu(K) \leq \mu(W) = \sup_{f \prec W} \lambda(f) \leq \lambda(g).$$
It follows that in Lemma 3.2, we have
$$\lambda(h_n) \leq (\sup h_n) \inf_{U \supset \operatorname{supp} h_n} \left(\sup_{g \prec U} \lambda(g)\right) = (\sup h_n)\, \mu(\operatorname{supp} h_n).$$

We check that μ is well-defined:

- μ **is an outer measure.** The untriviality lies in the countable subadditivity. By the definition of μ, it suffices to show the case for open sets. Given $U = \bigcup_j U_j$. For each $f \prec U$, by compactness, there exists N such that $f \prec \bigcup_{j=1}^N U_j$. There admits a partition of unity $\{\varphi_j\}_{j=1}^N$ for $\operatorname{supp} f$ subordinate to $\{U_j\}_{j=1}^N$, so that $\varphi_j f \prec U_j$ and

$$\lambda(f) = \sum_{j=1}^N \lambda(\varphi_j f) \leq \sum_{j=1}^N \mu(U_j).$$

- μ **is a Radon measure.** We use Lemma 2.2. R2 follows from the definition of μ. R3 is true if

$$\sup_{K \subset U} \mu(K) = \sup_{f \prec U} \lambda(f).$$

Indeed, given $K \subset U$, we have

$$\mu(K) = \inf_{g \succ K} \lambda(g) \leq \lambda(h)$$

for some $K \prec h \prec U$; and given $f \prec U$, we can find h and V such that

$$f \prec V \subset \overline{V} \prec h \prec U.$$

Then by Lemma 3.1,

$$\lambda(f) \leq \lambda(h) \Longrightarrow \lambda(f) \leq \inf_{h \succ \overline{V}} \lambda(h) = \mu\left(\overline{V}\right).$$

Now we check the **finite additivity of** μ **on compact sets**. Fix disjoint K_1, K_2, there exists $g \succ K_1 \cup K_2$ ε-approximating $K_1 \cup K_2$. We separate $\{K_i\}$ with disjoint $\{U_i\}$ and select $K_j \prec f_j \prec U_j$, then $f_j g \succ K_j$ and

$$\mu(K_1) + \mu(K_2) \leq \lambda(f_1 g) + \lambda(f_2 g) \leq \lambda(g) \leq \mu(K_1 \cup K_2) + \varepsilon \leq \mu(K_1) + \mu(K_2) + \varepsilon.$$

Finally, we show the integral expression of λ in terms of μ:

$$\lambda(f) = \int_X f \, \mathrm{d}\mu, \quad \forall f \in \mathcal{K}_+$$

by showing their difference being arbitrarily small.

Choose a partition of $\mathbb{R}_{\geq 0}$,

$$0 = t_0 < t_1 < \ldots < t_N < \infty, \quad \Delta t_j = t_j - t_{j-1} < \varepsilon.$$

We further require $t_N > \sup f > t_{N-1}$ and $\mu\left(x : f^{-1}(t_j)\right) = 0$, which is possible since

$$\mu(x : f > 0) \leq \mu(\operatorname{supp} f) < \infty.$$

Let $U_j = f^{-1}((t_{j-1}, t_j))$, $K = \operatorname{supp} f$. Then $\{U_j\}$ is a disjoint family of subsets of K with $\mu(K) \geq \sum_j \mu(U_j)$. Choose $K_j \subset U_j$ that $\frac{\varepsilon}{N}$-approximates U_j and $K_j \prec h_j \prec U_j$, then

$$\mu(U_j) - \frac{\varepsilon}{N} \leq \mu(K_j) \leq \lambda(h_j) \leq \mu(U_j).$$

Observe that

$$\left\{x : f > f\sum_j h_j, f(x) \neq t_i, \forall i\right\} \subset \bigcup_j (U_j \setminus K_j).$$

Since $\sup\left(f - f\sum_j h_j\right) \leq \sup f$, by Lemma 3.2 we have

$$\lambda\left(f\sum_j h_j\right) \leq \lambda(f) \leq \lambda\left(f\sum_j h_j\right) + \varepsilon\sup f.$$

Lastly, notice that trivially, $t_{j-1}h_j \leq fh_j \leq t_jh_j$. Taking sum from 1 to N and by the linearity of λ (using $t_{N-1} \leq \sup f$),

$$\sum_j t_{j-1}\mu(U_j) - \varepsilon\sup f \leq \lambda\left(f\sum_j h_j\right) \leq \sum_j t_j\mu(U_j).$$

Bridging the previous two results with $\lambda\left(f\sum_j h_j\right)$, we have

$$\sum_j t_{j-1}\mu(U_j) - \varepsilon\sup f \leq \lambda(f) \leq \sum_j t_j\mu(U_j) + \varepsilon\sup f.$$

Another similar approximation of $\int_X f\,\mathrm{d}\mu$ (using $\sup f \leq t_N$),

$$\sum_j t_{j-1}\mu(U_j) \leq \int_X f\,\mathrm{d}\mu = \int_{\bigcup_j U_j} f\,\mathrm{d}\mu \leq \sum_j t_j\mu(U_j),$$

together with the estimate of $\lambda(f)$ yields the desired result:

$$-\varepsilon(\mu(K) + \sup f) \leq \int_X f\,\mathrm{d}\mu - \lambda(f) \leq \varepsilon(\mu(K) + \sup f),$$

where $\sum_j (t_j - t_{j-1})\mu(U_j) \leq \varepsilon\mu(K)$. $\square$

For the second Riesz Representation Theorem, we use H to denote a finite-dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Set $\overline{\mathcal{K}} = C_c(X; H)$.

Theorem 3.2 (Riesz Representation Theorem No.2.) Suppose the linear functional $L : \overline{\mathcal{K}} \to \mathbb{R}$ is locally bounded. Then there is a unique Radon measure μ and μ-measurable function $\nu : X \to H$ with $|\nu| = 1$ μ-a.e. on X such that

$$L(f) = \int_X \langle f, \nu\rangle\,\mathrm{d}\mu, \quad \forall f \in \overline{\mathcal{K}}.$$

Remark L being locally bounded here means

$$\sup_{f \prec K} L(f) < \infty, \quad \forall K \subset X,$$

where $f = \sum_j f_je_j \prec K$ means $f_j \prec K$ for all j. Local boundedness has to be imposed here because no monotonicity result can be derived without L being non-negative.

Proof. It suffices to take $H=\mathbb{R}^n$ using an orthonormal basis $\{e_i\}$. Define

$$\lambda(f)=\sup_{|\omega|\le f} L(\omega), \quad \forall f\in\mathcal{K}_+.$$

It is expected that $\lambda(f)=\int_X f\,\mathrm{d}\mu$ by the statement of the theorem. Let's check the conditions in Theorem 3.1 for λ. It's clear that $\lambda(cf)=c\lambda(f)$ for all $c\ge 0$. Next we show

$$\lambda(f+g)=\lambda(f)+\lambda(g).$$

Since $|\omega_1|\le f$ combined with $|\omega_2|\le g$ imply $|\omega_1+\omega_2|\le f+g$, we have $\lambda(f+g)\ge\lambda(f)+\lambda(g)$. The reverse is trickier. For $|\omega|\le f+g$, define

$$\omega_1=\frac{f}{f+g}\omega\cdot\mathbf{1}_{\{f+g>0\}}, \quad \omega_2=\frac{g}{f+g}\omega\cdot\mathbf{1}_{\{f+g>0\}}.$$

Then $\omega_1+\omega_2=\omega$, $|\omega_1|\le f$, $|\omega_2|\le g, \omega_i\in\overline{\mathcal{K}}$, implying the reverse. Hence, by Theorem 3.1,

$$\lambda(f)=\sup_{|\omega|\le f} L(\omega)=\int_X f\,\mathrm{d}\mu, \quad \forall f\in\mathcal{K}_+$$

for a Radon measure μ on X.

Now write $f=\sum_j f_j e_j$ in $\overline{\mathcal{K}}$, then

$$L_j(f_j):=L(f_j e_j)\le\sup_{|\omega|\le|f_j|} L(\omega)=\int_X |f_j|\,\mathrm{d}\mu=\|f_j\|_{L^1(\mu)}.$$

We apply the *Riesz Representation Theorem for $L^1(\mu)$* to any $K\subset X$ to obtain a globally well-defined bounded μ-measurable function ν_j such that $L_j(f_j)=\int_X f_j\nu_j\,\mathrm{d}\mu$; that is,

$$L(f)=\int_X f\cdot\nu\,\mathrm{d}\mu, \quad \forall f\in\overline{\mathcal{K}},$$

where $\nu=\sum_j\nu_j e_j$. It remains to check that $|\nu|=1$ μ-a.e. The above derivation suggests

$$\int_X f\,\mathrm{d}\mu=\sup_{|g|\le f} L(g)\le\int_X f|\nu|\,\mathrm{d}\mu, \quad \forall f\in\mathcal{K}_+.$$

We claim that equality holds, so by Lemma 2.3 we finish the proof. Take $K=\operatorname{supp} f$, then the function $\operatorname{sgn}\nu\cdot\mathbf{1}_K\in L^1$, which can be approximated in L^1 by some $\{h_n\}\subset\overline{\mathcal{K}}$ by Lemma 2.3. The shrinking map

$$\tau: L^1(X;H)\to L^1(X;H), \quad \varphi\mapsto\operatorname{sgn}\varphi\cdot\min\{1,|\varphi|\}$$

ensures that we may require $|h_n|\le\mathbf{1}_K$, as

$$\|\tau(h_n)-\tau(\operatorname{sgn}\nu\cdot\mathbf{1}_K)\|_{L^1}\le\|h_n-\operatorname{sgn}\nu\cdot\mathbf{1}_K\|_{L^1}\to 0, \quad n\to\infty,$$

where $|\tau(h_n)|\le\mathbf{1}_K$ and $\tau(\operatorname{sgn}\nu\cdot\mathbf{1}_K)=\operatorname{sgn}\nu\cdot\mathbf{1}_K$. Finally, by taking $g_n=fh_n$,

$$\begin{aligned}\left|\int_X g_n\cdot\nu\,\mathrm{d}\mu-\int_X f|\nu|\,\mathrm{d}\mu\right| &\le\int_X f\,|h_n\cdot\nu-|\nu||\,\mathrm{d}\mu\\ &\le(\sup f)(\sup|\nu|)\int_{K_0}|(h_n-\operatorname{sgn}\nu)\cdot\operatorname{sgn}\nu|\,\mathrm{d}\mu\\ &\le(\sup f)(\sup|\nu|)\|h_n-\operatorname{sgn}\nu\|_{L^1(K_0)}\to 0, \quad n\to\infty,\end{aligned}$$

where $K_0=K\cap\operatorname{supp}\nu$. $\square$

Remark We required ν_j to be local because the Riesz Representation Theorem for L^1,

$$\left(L^1(\mu)\right)^* = L^\infty(\mu),$$

holds if μ is σ-finite, and our μ is indeed σ-finite, actually finite, locally speaking. Different ν_j are compatible with each other in different compact sets μ-a.e.

5 Compactness Theorem For Radon Measures

If X is a normed vector space, then the closed unit ball in X or X^* is compact if and only if X is finite-dimensional. But we always obtain compactness under a weaker topology:

Lemma 4.1 (Banach-Alaoglu) If X is a normed vector space, then the closed unit ball in X^* is compact in the weak* topology.

Remark:

- The *weak topology* on X is the coarsest topology such that all $f \in X^*$ are continuous.
- The *weak* topology* on X^* is the coarsest topology such that all $x \in X \hookrightarrow X^{**}$ are continuous as functionals on X^*.
- An important special case is when X is *separable*, then the closed unit ball in X^* is sequentially compact, as the weak* topology is *metrizable* in this case. A legitimate metric is defined by

  $$\operatorname{dist}(\varphi, \psi) = \sum_j \frac{1}{2^j} \cdot \frac{|\varphi(x_i) - \psi(x_i)|}{1 + |\varphi(x_i) - \psi(x_i)|}, \quad \forall\, \varphi, \psi \in X^*,$$

  where $\{x_i\}$ is a countable dense subset of X.

We expand our previous notations as follows:

$$\mathcal{K} = C_c(X), \quad \mathcal{K}^+ = C_c^+(X), \quad \mathcal{K}_j = C_c(K_j), \quad \mathcal{K}_j^+ = C_c^+(K_j).$$

Theorem 4.1 (Compactness Theorem for Radon Measures) Suppose $\{\mu_k\}$ is a sequence of Radon measures on a *σ-compact* LCH space X that is locally bounded, i.e.

$$\sup_k \mu_k(K) < \infty, \quad \forall\, K \subset X.$$

Then $\{\mu_k\}$ converges to a Radon measure μ weakly up to a subsequence. That is, there exists a subsequence $\{\mu_{k_j}\}$ s.t.

$$\int_X f \,\mathrm{d}\mu_{k_j} \longrightarrow \int_X f \,\mathrm{d}\mu, \quad j \to \infty, \quad \forall\, f \in \mathcal{K}.$$

Remark In the following proof, $F_{j,k} \in \mathcal{K}_j^*$ has operator norm equals $\mu_k(K_j)$, so for each j, $\{F_{j,k}\}_k$ is uniformly bounded. That is how we leverage Lemma 4.1.

Proof. By σ-compactness, there is an increasing sequence $\{K_j\}$ with $X = \bigcup_j K_j$. Using Lemma 1.1, we further require $K_j \subset \operatorname{int} K_{j+1}$, then $X = \bigcup_j K_j = \bigcup_j \operatorname{int} K_j$. (Such $\{K_j\}$ is called an *exhaustion* of X by compact sets.) Set

$$F_{j,k} : \mathcal{K}_j \to \mathbb{R}, \quad f \mapsto \int_{K_j} f \, \mathrm{d}\mu_k.$$

Fix j. Since $\mathcal{K}_j$ is separable, by Lemma 4.1, there is a subsequence of $\{F_{j,k}\}_k$ convergent to some $F_j \in \mathcal{K}_j^*$ in the weak sense. Without loss of generality, we do not have to take any subsequences for the above convergence, by repetitively selecting subsequences and taking the diagonal indices. Observe that each $F_{j,k}$ is non-negative,

$$F_j\left(f\right) = \lim_{k\to\infty} F_{j,k}\left(f\right) \geq 0, \quad \forall\, f \in \mathcal{K}_j^+,$$

so Theorem 3.1 ensures that $F_j\left(f\right) = \int_{K_j} f \, \mathrm{d}\nu_j$ for a Radon measure ν_j on K_j and $f \in \mathcal{K}_j$. Notice that $\{\nu_j\}$ is actually compatible, because $\{\mathcal{K}_j\}$ is increasing, and for each $l > j$,

$$\int_{K_j} f \, \mathrm{d}\nu_j = \lim_{k\to\infty} \int_{K_j} f \, \mathrm{d}\mu_k = \lim_{k\to\infty} \int_{K_l} f \, \mathrm{d}\mu_k = \int_{K_l} f \, \mathrm{d}\nu_l, \quad \forall\, f \in \mathcal{K}_j,$$

which means $\nu_j = \nu_l|_{K_j}$. We next define

$$F : \mathcal{K} \to \mathbb{R}, \quad f \mapsto \lim_{j\to\infty} \int_{K_j} f \, \mathrm{d}\nu_j < \infty,$$

which is easily checked to be linear and non-negative. It is finite because for j large enough, $\operatorname{supp} f \subset \operatorname{int} K_j$. Theorem 3.1 again gives a Radon measure μ on X such that $F\left(f\right) = \int_X f \, \mathrm{d}\mu$ for all $f \in \mathcal{K}$. It is compatible with all ν_j. If we choose $k = k_j$ such that

$$\left|F_{j,k_j}\left(f\right) - F_j\left(f\right)\right| < \frac{1}{j},$$

then $F_{j,k_j}\left(f\right) \to F\left(f\right)$ as $j \to \infty$. Finally, for j large enough,

$$F_{j,k_j}\left(f\right) = \int_{K_j} f \, \mathrm{d}\mu_{k_j} = \int_X f \, \mathrm{d}\mu_{k_j} \to \int_X f \, \mathrm{d}\mu, \quad \forall\, f \in \mathcal{K},$$

completing the proof. $\square$

References

[1] Leon Simon, *Lectures on Geometric Measure Theory*, Australian National University, 1983.

[2] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications (Second Edition)*, John Wiley & Sons, Inc., 1999.