Monotonicity of Lebesgue Spaces

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We know that on $\mathbb{R}$ or $\mathbb{R}_{>0}$, Lebesgue spaces do not have rigorous monotonicity, but they do have on $[0,1]$ or finite sets with counting measure. Here is the formal generalization:

Theorem (Monotonicity of Lebesgue Spaces). Let $(X, \mathcal{M}, \mu)$ be a measure space.

- Decreasing Property: There exists $1 \leq p < q < \infty$ such that
$$L^q(X, \mathcal{M}, \mu) \subset L^p(X, \mathcal{M}, \mu)$$
if and only if
$$\sup\{\mu(E) : E \in \mathcal{M} \setminus \mathcal{I}\} < \infty,$$
where $\mathcal{I}$ denotes subsets of infinite measure.

- Increasing Property: There exists $1 \leq p < q < \infty$ such that
$$L^p(X, \mathcal{M}, \mu) \subset L^q(X, \mathcal{M}, \mu)$$
if and only if
$$\inf\{\mu(E) : E \in \mathcal{M} \setminus \mathcal{N}\} > 0,$$
where $\mathcal{N}$ denotes subsets of zero measure, or null sets.

Proof. Notations for the following two parts are independent. Set $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

1. $\Longrightarrow$ Suppose first the finite measures are bounded by $C < \infty$. Take any $f \in L^q$, then f is supported on $E = \bigcup_k E_k$ for
$$E_k = \left\{x \in X : |f| > \frac{1}{k}\right\}, \quad \forall\, k \in \mathbb{N}^*.$$
By Chebyshev's inequality,
$$\mu(E_k) \leq \left(\frac{\|f\|_{L^q}}{1/k}\right)^q < \infty, \quad \forall\, k \in \mathbb{N}^*,$$
so by assumption, $\mu(E_k) \leq C$. Since $\{E_k\}$ is increasing, $\mu(E) = \lim_{k\to\infty} \mu(E_k) \leq C$. By Holder's inequality,
$$\int_E |f|^p \cdot 1 \mathrm{d}\mu \leq \left(\int_E (|f|^p)^{\frac{q}{p}} \mathrm{d}\mu\right)^{\frac{p}{q}} \left(\int_E 1^{\frac{q}{q-p}} \mathrm{d}\mu\right)^{\frac{q-p}{q}} = \mu(E)^{\frac{q-p}{q}} \left(\int_E |f|^q \mathrm{d}\mu\right)^{\frac{p}{q}} < \infty.$$

Hence, $L^q \subset L^p$ with

$$\|f\|_{L^p} \leq C^{\frac{1}{r}} \|f\|_{L^q}, \quad \forall f \in L^q.$$

$\Longleftarrow$ Conversely, suppose $L^q \subset L^p$. Consider the inclusion map $\iota : L^q \hookrightarrow L^p$. We will use the *Closed Graph Theorem* to show that ι is bounded. Indeed, given a Cauchy sequence $\{(f_n, f_n)\}$ in $L^q \times L^p$, $\{f_n\}$ is both Cauchy in L^q and L^p. By the completeness of Lebesgue spaces,

$$f_n \to \bar{f} \quad \text{in } L^q, \quad f_n \to \widetilde{f} \quad \text{in } L^p$$

for some $\bar{f} \in L^q$ and $\widetilde{f} \in L^p$. Take a subsequence $\{f_{n_k}\}$ convergent to $\bar{f}$ a.e., and a subsubsequence of which convergent to $\widetilde{f}$ a.e. By the uniqueness of almost-everywhere convergence, $\bar{f} = \widetilde{f}$ a.e. Hence $(f_n, f_n) \to \left(\bar{f}, \widetilde{f}\right)$ in $L^q \times L^p$. The graph of ι is closed thus bounded.

Take $h = \chi_A$ for an arbitrary $A \in \mathcal{M} \setminus \mathcal{I}$, we have $h \in L^p \cap L^q$ and

$$\mu(A)^{\frac{1}{p}-\frac{1}{q}} = \frac{\mu(A)^{\frac{1}{p}}}{\mu(A)^{\frac{1}{q}}} = \frac{\|h\|_{L^p}}{\|h\|_{L^q}} \leq \sup_{0 \neq g \in L^q} \frac{\|g\|_{L^p}}{\|g\|_{L^q}} = \|\iota\| < \infty,$$

so that $\mu(A) \leq \|\iota\|^r < \infty$.

2. $\Longrightarrow$ Now suppose the positive measures have a lower bound $\alpha > 0$. Assume without lose of generality that $\varphi \in L^p \setminus \{0\}$, we interpolate in terms of the L^∞ norm. For each $0 < t < \|\varphi\|_{L^\infty}$, by the definition of L^∞ norms, the set

$$F_t = \{x \in X : |\varphi| > t\}$$

has positive measure. By Chebyshev's inequality, arguing as in part 1, its measure is finite. According to the assumption, $\mu(F_t) \geq \alpha$. We have

$$\|\varphi\|_{L^p} \geq \left(\int_{F_t} |\varphi|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \geq t\mu(F_t)^{\frac{1}{p}} \geq t\alpha^{\frac{1}{p}}.$$

As t was taken arbitrarily, $\|\varphi\|_{L^\infty} \leq \|\varphi\|_{L^p} \alpha^{-\frac{1}{p}}$. Returning to the L^q norm,

$$\|\varphi\|_{L^q} = \left(\int_X |\varphi|^p |\varphi|^{q-p} \, \mathrm{d}\mu\right)^{\frac{1}{q}} \leq \|\varphi\|_{L^\infty}^{\frac{q-p}{q}} \|\varphi\|_{L^p}^{\frac{p}{q}} \leq \alpha^{-\frac{1}{r}} \|\varphi\|_{L^p} < \infty.$$

Hence, $L^p \subset L^q$ with

$$\|\varphi\|_{L^q} \leq \alpha^{-\frac{1}{r}} \|\varphi\|_{L^p}, \quad \forall \varphi \in L^p.$$

$\Longleftarrow$ Conversely, suppose $L^p \subset L^q$. Arguing as before, we find that the inclusion map $\iota : L^p \hookrightarrow L^q$ is bounded. For any $B \in \mathcal{M} \setminus (\mathcal{I} \cup \mathcal{N})$, $h = \chi_B$ belongs to $L^p \cap L^q$. We have

$$\mu(B)^{\frac{1}{q}-\frac{1}{p}} = \frac{\mu(B)^{\frac{1}{q}}}{\mu(B)^{\frac{1}{p}}} = \frac{\|h\|_{L^q}}{\|h\|_{L^p}} \leq \sup_{0 \neq g \in L^p} \frac{\|g\|_{L^q}}{\|g\|_{L^p}} = \|\iota\| < \infty,$$

so that $\mu(B) \geq \|\iota\|^{-r} > 0$.

$\square$