The Gaussian Curvature Formula for Graphs of Functions on $\mathbb{R}^2$

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July 2026

For $f \in C^2(\Omega; \mathbb{R})$, a twice continuously differentiable function on an open set $\Omega \subset \mathbb{R}^2$, we present a computation of the Gaussian curvature κ_p of the graph Σ of f above any $p = (x_0, y_0) \in \Omega$. Since Σ has a globally defined unit normal N, κ_p is defined as the determinant of the linear transform

$$\nabla N : T_p\Sigma \to T_p\Sigma, \quad v \mapsto \nabla_v N,$$

where $T_p\Sigma$ is the tangent plane of Σ at the point above p, and $\nabla_v N$ is the directional derivative of N along v. Recall that we may choose an arbitrary smooth curve $r : (-\varepsilon, \varepsilon) \to \Sigma$ with $r(0) = p$, $r'(0) = v$, and always have

$$\nabla_v N = (N \circ r)'(0).$$

A pair of transverse curves gives a basis for $T_p\Sigma$. We take

$$\begin{aligned} r_1 &: (-\varepsilon, \varepsilon) \to \Sigma, \quad x \mapsto (x + x_0, y_0, f(x + x_0, y_0)), \\ r_2 &: (-\varepsilon, \varepsilon) \to \Sigma, \quad y \mapsto (x_0, y + y_0, f(x_0, y + y_0)), \end{aligned}$$

and have

$$e_1 = r_1'(0) = (1, 0, f_x), \qquad e_2 = r_2'(0) = (0, 1, f_y),$$

if we embed $T_p\Sigma$ into $\mathbb{R}^3$ canonically. Let

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto (x, y)$$

be the orthogonal projection onto the first two components; then $\{\pi(e_1), \pi(e_2)\}$ is the standard orthonormal basis of $\mathbb{R}^2$, which turns out to simplify the computations significantly. By taking the upward-pointing unit normal, we have

$$N_p = \frac{e_1 \times e_2}{|e_1 \times e_2|} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + |\nabla f|^2}}.$$

The next job is to compute the two directional derivatives $\nabla_{e_1} N$, $\nabla_{e_2} N$. Using the two curves r_1, r_2, we have the relations

$$\nabla_{e_1} N = (N \circ r_1)'(0) = N_x, \qquad \nabla_{e_2} N = (N \circ r_2)'(0) = N_y,$$

where N_x and N_y are the partial derivatives of N at p in the x and y directions, respectively. Since $N_x, N_y \in T_p\Sigma = \operatorname{span}(e_1, e_2)$, there exist coefficients $a, b, c, d \in \mathbb{R}$ such that

$$N_x = ae_1 + be_2, \qquad N_y = ce_1 + de_2.$$

Under the projection π, we immediately find that the Jacobian matrix of ∇N in the basis $\{e_1, e_2\}$ equals

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} N_x^1 & N_y^1 \\ N_x^2 & N_y^2 \end{bmatrix},$$

where

$$N_x = (N_x^1, N_x^2, N_x^3), \qquad N_y = (N_y^1, N_y^2, N_y^3).$$

Lastly, we compute N_x^1, N_x^2, N_y^1, N_y^2. By denoting

$$w = \nabla f = \begin{bmatrix} f_x & f_y \end{bmatrix},$$

we have

$$\begin{aligned}
N_x^1 &= \left(-\frac{f_x}{\sqrt{1+ww^T}}\right)_x \\
&= -\frac{f_{xx}\sqrt{1+ww^T} - f_x \frac{2ww_x^T}{2\sqrt{1+ww^T}}}{1+ww^T} \\
&= \left(1+ww^T\right)^{-\frac{3}{2}} \left(-f_{xx}(1+ww^T) + f_x ww_x^T\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
N_x^1 &= \left(1+ww^T\right)^{-\frac{3}{2}} \left(-f_{xx}(1+ww^T) + f_x ww_x^T\right), \\
N_x^2 &= \left(1+ww^T\right)^{-\frac{3}{2}} \left(-f_{xy}(1+ww^T) + f_y ww_x^T\right), \\
N_y^1 &= \left(1+ww^T\right)^{-\frac{3}{2}} \left(-f_{xy}(1+ww^T) + f_x ww_y^T\right), \\
N_y^2 &= \left(1+ww^T\right)^{-\frac{3}{2}} \left(-f_{yy}(1+ww^T) + f_y ww_y^T\right).
\end{aligned}$$

As a result,

$$\begin{aligned}
\kappa_p &= N_x^1 N_y^2 - N_x^2 N_y^1 \\
&= \left(1+ww^T\right)^{-3} \begin{vmatrix} -f_{xx}(1+ww^T) + f_x ww_x^T & -f_{xy}(1+ww^T) + f_y ww_x^T \\ -f_{xy}(1+ww^T) + f_x ww_y^T & -f_{yy}(1+ww^T) + f_y ww_y^T \end{vmatrix} \\
&= \left(1+ww^T\right)^{-3} \left(\begin{matrix} \begin{vmatrix} -f_{xx}(1+ww^T) & -f_{xy}(1+ww^T) \\ -f_{xy}(1+ww^T) & -f_{yy}(1+ww^T) \end{vmatrix} + \begin{vmatrix} f_x ww_x^T & f_y ww_x^T \\ -f_{xy}(1+ww^T) & -f_{yy}(1+ww^T) \end{vmatrix} \\ + \begin{vmatrix} -f_{xx}(1+ww^T) & -f_{xy}(1+ww^T) \\ f_x ww_y^T & f_y ww_y^T \end{vmatrix} + \begin{vmatrix} f_x ww_x^T & f_y ww_x^T \\ f_x ww_y^T & f_y ww_y^T \end{vmatrix} \end{matrix} \right) \\
&= \left(1+ww^T\right)^{-3} \left(\begin{matrix} (1+ww^T)^2 \left|\mathrm{Hess} f\right| \\ + (1+ww^T) \begin{bmatrix} f_x & f_y \end{bmatrix} \left(\begin{vmatrix} f_x & f_y \\ -f_{xy} & -f_{yy} \end{vmatrix} \begin{bmatrix} f_{xx} \\ f_{xy} \end{bmatrix} + \begin{vmatrix} -f_{xx} & -f_{xy} \\ f_x & f_y \end{vmatrix} \begin{bmatrix} f_{xy} \\ f_{yy} \end{bmatrix} \right) \end{matrix} \right) \\
&= \left(1+ww^T\right)^{-3} \left((1+ww^T)^2 \left|\mathrm{Hess} f\right| - (1+ww^T)\, w \left|\mathrm{Hess} f\right| w^T\right) \\
&= \left(1+ww^T\right)^{-3} \left((1+ww^T)^2 - (1+ww^T)ww^T\right) \left|\mathrm{Hess} f\right| \\
&= \frac{\left|\mathrm{Hess} f\right|}{(1+ww^T)^2}.
\end{aligned}$$

Here $|\mathrm{Hess} f|$ denotes the determinant of the Hessian matrix. We conclude that

$$\kappa_p = \frac{\left|\mathrm{Hess} f\right|}{\left(1 + |\nabla f|^2\right)^2}.$$

In particular, at critical points, where $\nabla f = 0$, the Gaussian curvature equals the determinant of the Hessian matrix.